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AN ELEMENTARY TREATISE

ON THE

DIFFERENTIAL CALCULUS

FOUNDED ON THE

METHOD OF RATES

BY
WILLIAM WOOLSEY JOHNSON
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PREFACE.

In the year 1879, I published in connection with the late Professor J. M. Rice, an *Elementary Treatise on the Differential Calculus, founded on the Method of Rates or Fluxions*. In the present work, the text is essentially new, and the order of the subjects will be found to differ in many points from that of the older book. I still employ the Newtonian conception of rates, and the functional method of obtaining the derivative, notably in the case of the logarithmic function, page 49; but the method is more closely connected with that of limits, and an independent basis for the algebraic functions is established in Art. 31.

The differential of a variable not a linear function of the time is treated as an hypothetical increment corresponding to an assumed increment or differential of time, dt, page 18. It is assumed that there exists at any instant a definite measure of a rate, although it may be variable. In other words, the variables dealt with being assumed continuous, their rates are also assumed to admit of a continuously varying measure. In velocity, the graphic form of rate, these hypothetical increments receive a beautiful realization in the use of Attwood's machine, see Art. 21. The variable t (when it stands for elapsing time) is thus taken as the natural independent variable incapable of a non-uniform rate simply because, from its fundamental character, it is the standard to which all other variables are referred when we speak of their rates of variation.

With this interpretation of dx, its ratio to dt is independent of the value of dt, and is the measure of the rate of x when a definite function of t. In the present work, the ratio of actual increments is at once shown to have the rate as its limit, and the evanescent quantity e expressing the difference is used in a general formula, Art. 26, for the non-linear function of t. From this formula the differential of the product, Art. 31, and thence those of the algebraic functions are readily deduced.

When x is an independent variable, the derivative of a function y is the measure of its relative rate, and the differentials become simultaneous hypothetical increments. The independence of their ratio upon their actual magnitudes is thus identified with the existence of a definite tangent to the curve which is the graph of the function.

This notion leads naturally, in the case of the circular functions, to a mode of differentiation founded directly upon the geometrical definitions by which they are first introduced to the student, and not upon analytical properties subsequently deduced.

The advantage of giving to dx and dy separate meanings, and finite magnitudes no longer requiring continuous reference to the limit, is apparent in the mode of statement of the formulæ of differentiation and the consequent treatment of the function of a function, Art. 48.

Chapter VI presents a full treatment of the subject of development in power-series. In the text and answers to the examples will be found the developments of nearly all the elementary functions, including the direct and inverse hyperbolic functions.

In the pages devoted to curves and curve tracing, the more essential algebraic methods as well as those peculiar to the Differential Calculus are exemplified. These pages will be found to contain a fairly full treatment of the higher plane curves most commonly met with. Numerous entries in the alphabetical index are given to facilitate reference to the results established.

The topics have been so arranged that sections X., XII., XV., XVIII., XX., XXI., and XXXIII. can be omitted to form a shorter course to be supplemented by selections from the applications to curves and by the first section of the final chapter.

W. WOOLSEY JOHNSON.

June, 1904.

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THE
DIFFERENTIAL CALCULUS.

CHAPTER I.

FUNCTIONS, RATES AND DERIVATIVES.

I.

Continuous Variables.

1. A MAGNITUDE of any sort which can be measured by a unit of its own kind is represented by the number of such units or parts of a unit which it contains. This number is called its ***numerical measure.*** If the magnitude is capable of indefinite subdivision, the corresponding numerical measure is conceived of as capable of an unlimited number of values. Such a numerical value, or "number," may be regarded as passing gradually from one fixed value to another in a certain interval of time, in such a manner as to assume during the interval every intermediate value; it is then called a ***continuous variable.*** For example, if a point P is moving along a fixed straight line, its distance OP from a fixed point O taken on the line is a continuous variable.

The variables with which we have to deal in the Differential Calculus are always abstract quantities, or numerical values independent of any particular kind of unit, but it is frequently convenient to represent them graphically by linear distances.

Functions.

2. A quantity which depends for its value upon another quantity is said to be a *function* of it. A function of a continuous variable will itself be a continuous variable, and the variable upon which it depends is in distinction called *the independent variable.* Denoting the independent variable by x, an algebraic or trigonometric expression containing x is generally a function of x, thus x^2 is a function of x; but the expressions x^0, $x^2 + (a - x)(a + x)$, $(\tan x + \cot x)\sin 2x$, are not functions of x, since each admits of expression in a form which does not contain x.

3. In treating of the general properties of functions, it is necessary to have a symbol which may denote any function of x. The symbol usually employed for this purpose is $f(x)$, and when several functions occur in the same investigation, such symbols as $F(x)$, $F'(x)$, $\phi(x)$, etc., are used, the enclosed letter in each case being the independent variable. Supposing the meaning of the functional symbol f in a given case to be defined, the value of the function corresponding to a particular value of x is expressed by substituting this value for x in the symbol $f(x)$. Thus, if we are given

$$f(x) = x^3 - 2x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

we have

$$f(1) = -1, \qquad f(2) = 4, \qquad f(-1) = 1, \qquad f(0) = 0.$$

Here equation (1) defines the meaning of f, and then the other equations follow; but none of these last equations serve, either alone or in combination, to define f.

Again, if we are given

$$F(x) = \log_a x,$$

where a denotes the base of a system of logarithms, we have

$$F(1) = 0, \quad F(0) = -\infty, \quad F(a) = 1.$$

4. If a quantity which is independent of the independent variable x occurs in the expression for a function, it is, in contradistinction, called a *constant*, even when it has not an absolute value, but is represented (like a in the example above) by a letter to which any value may be assigned. Thus, when a^x is regarded as a function of x and denoted by $f(x)$, a is called a constant. When it is desired to put in evidence the fact that such a quantity is susceptible of different values, it is called a variable independent of x; the function then depends upon *two independent variables*. In this case, both variables are enclosed between the marks of parenthesis. Thus we may write

$$f(x, a) = a^x;$$

and, taking this equation to define f as a function of two independent variables, we have

$$f(y, b) = b^y, \quad f(3, 2) = 8, \quad f(2, 3) = 9.$$

Implicit Functions.

5. When an equation is given involving two variables, x and y, either variable may be regarded as independent and the other as a function of it. If x is the independent variable, and y is not directly expressed in terms of x, it is called an *implicit function* of x. Thus, if we have

$$ax^2 - 3axy + y^3 - a^3 = 0,$$

y is an implicit function of x, or is said to be a function of x *given in the implicit form.* Again, x is by virtue of the equation an implicit function of y. But, if we solve the equation for x, we obtain

$$x = \frac{3y}{2} \pm \sqrt{\left(a^2 + \frac{9y^2}{4} - \frac{y^3}{a}\right)},$$

and in this form, x is, in contradistinction, called an *explicit function* of y.

The Graph of a Function.

6. When a function is given in the explicit form

$$y = f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

it is often useful to construct the curve of which (1) is the equation as a graphic illustration of the mode in which the function or ordinate varies with the change of the independent variable or abscissa. For this purpose, rectangular coordinates are always employed, and the curve is called *the graph of the function.* For example, Fig. 1 shows the curve whose equation is $y = x^2$. This curve, which is a parabola, is therefore the graph of the function x^2.

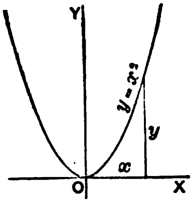

FIG. 1.

Inverse Functions.

7. If an equation between x and y be solved in each of the forms

$$y = f(x) \qquad \text{and} \qquad x = \phi(y),$$

each of the functions f and ϕ is said to be *the inverse function* of the other. Thus, if $y = x^2$, $x = \pm \sqrt{y}$; the function "square-root" is therefore the inverse of the function "square." The graph of the inverse of a given function is the same curve as that of the given function in another position. Thus, Fig. 1 is the graph of the function square-root if we regard y as the independent variable, and Fig. 2, p. 6, is the graph when x is the independent variable. The square-root is a *two-valued* function; but, since we can distinguish between the two values by means of their signs, the symbol $\sqrt{x}$ may be regarded as a one-valued function.

In the case of the trigonometric functions, a peculiar notation for the inverse functions has been employed. Thus if

$$x = \sin\theta, \qquad \text{we write} \qquad \theta = \sin^{-1}x.$$

Symbols of this character are, in the Calculus, always taken to denote the *arcual measures* of the angles in question or ratio of the arc to the radius, of which the unit is called *the radian.* For example, because the sine of 30° is $\frac{1}{2}$, and its arcual measure $\frac{1}{6}\pi$, we may write $\frac{1}{6}\pi = \sin^{-1}\frac{1}{2}$. The inverse trigonometric functions are all many-valued functions, and the mode of distinguishing between the different values will be considered later.

Continuity of a Function.

8. A one-valued function of a continuous variable is itself a continuous variable, at least for certain ranges of values of the independent variable. For example, the function x^2 is continuous for all values of x; in other words, if $y = x^2$, as x passes from any one given finite value to any other, y passes *gradually*, and not by any sudden leaps, from its first to its last value. Accordingly, the graph of the function, that is the curve $y = x^2$, is a continuous line from the point for which

x has any given value a to that at which $x = b$, as illustrated by Fig. 1.

On the other hand, the function square-root is continuous only for positive values of the independent variable. Thus, in Fig. 2, which is the graph of $y = \sqrt{x}$, considered as a one-valued function of x, the curve stops abruptly at the point where $x = 0$, since there are no real values of the function for negative values of x. But the curve is continuous between the points for which x has any two values >0. In this case, the function is said to become imaginary for values <0.

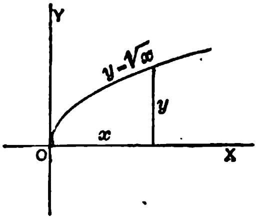

FIG. 2.

9. Again, there may be values of the independent variable x in approaching which the value of the function y increases without limit. For example, the function

$$y = \frac{1}{x - 1}$$

has a large value when x is a little greater than 1, and increases without limit as x approaches nearer to 1. The function is, therefore, said to become *infinite* when $x = 1$. This function has a negative value for all values of $x<1$, and approaches $-\infty$ when x approaches 1. It is said to be *discontinuous* for any range of values which includes the value $x = 1$. Accordingly, the graph of the function, Fig. 3, consists of two branches which do not form a continuous line, so

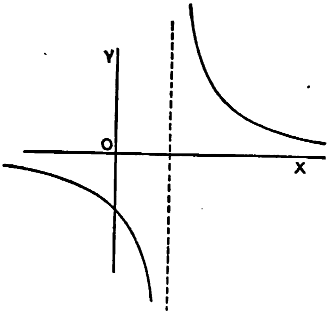

FIG. 3.

that we cannot pass continuously from a value of the function for which x is algebraically less than unity to one for which $x > 1$.

10. As a further illustration, the function

$$y=\sqrt{\frac{x+1}{x}} \quad . \ . \ . \ . \ . \ . \ . \quad (1)$$

is real for all positive values of x; but, when x approaches zero, it becomes infinite. Moreover it is imaginary for values of x between zero and -1. But from $x = -1$ to $x = -\infty$, the function is again real. Thus it is continuous from $x = 0$ to $x = \infty$, and also from $x = -1$ to $x = -\infty$. Accordingly the graph, shown in Fig. 4, consists of two branches which are discontinuous one with the other. It will be noticed that the equation

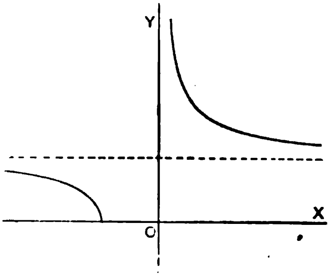

FIG. 4.

$$xy^2 = x + 1, \quad . \ . \ . \ . \ . \ . \ . \quad (2)$$

found by rationalizing equation (1), represents also the graph of the other value (which is negative) of the two-valued function given in the implicit form by equation (2). Again, the complete curve, consisting of three branches, forms the graph of the function inverse to that considered, namely

$$x = \frac{1}{y^2 - 1}.$$

This last is a one-valued function, and suffers discontinuity (by becoming infinite) when $y = 1$, and also when $y = -1$.

Increasing and Decreasing Functions.

11. A function is said to be an *increasing* one when its value *increases with the increase of the independent variable.* Such a function necessarily decreases with the decrease of the independent variable. On the other hand, a function is a *decreasing* one when it *decreases with the increase* of the independent variable, and consequently increases with the decrease of that variable.

The same function may of course be an increasing one for a certain range of values of the independent variable and a decreasing one for another range. Thus sin x is an increasing function while x passes from 0 to $\frac{1}{2}\pi$; it is then a decreasing one while x passes from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, since for this range of values of x its value decreases from its greatest value $+1$ to its least value algebraically considered, namely -1.

12. When the graph of the function is drawn, algebraic increase of the independent variable x means motion from left to right, and algebraic increase of y means motion upward. Hence, supposing a point to describe the graph moving toward the right, it will move upward if the function is an increasing one. In this case, the *slope* of the curve is said to be *positive.* In the opposite case, representing a decreasing function, the slope is *negative.* Thus in Fig. 1, the graph of the function x^2, the slope is positive and the function is an increasing one for all positive values of x. But the slope is negative and the function is a decreasing one for all negative values of x, since y decreases as these values algebraically increase.

Again, Fig. 2 shows that the positive value of the inverse of this function is an increasing function. Fig. 3 has a negative slope throughout both branches, so that $\dfrac{1}{x-1}$ is always a decreasing function. In like manner, Fig. 4 shows that $\sqrt{\dfrac{x+1}{x}}$ is, when real, always a decreasing function of x.

The Linear Function.

13. The simplest of all functions is the rational algebraic expression containing x only in the first degree. Its general form is

$$y = mx + b,$$

where m and b are constants. The corresponding graph is therefore an oblique straight line, and for this reason the function is said to be of the *linear* form. The linear function is continuous for all values of x, and is throughout either an increasing one or a decreasing one, according as m is positive (as in Fig. 5) or negative. The slope of the graph is constant, and m (the tangent of the angle it makes with the axis of x, or ratio of the rise to the corresponding horizontal distance) is taken as its measure, and is called the *gradient*.

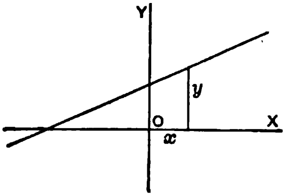

FIG. 5.

14. In the case of all other functions, the graph has a variable slope. If the value of the function y can be expressed either rationally or fractionally in terms of powers of x, or, in the case of implicit functions, if the relation between x and y is algebraic, so that the graph is an algebraic curve, the function is said to be an *algebraic* one. All other functions are called *transcendental.*

Functional Equations.

15. An equation involving an unknown function, that is to say, the values of such a function corresponding to different values of the variable upon which it depends, is called a *functional equation.* The values of the variable may either be connected or independent of one another. In either case, the equation expresses a *property* of the function. Thus $f(x) = f(-x)$ expresses a property of the function symbolized for the present by f. The information conveyed by this equation does not go far toward determining f, because the property is shared by a great variety of functions, such as $\cos x$, $x^4 + x^2$, e^{x^2}, etc.

On the other hand, such an equation as

$$f(xy) = f(x) + f(y),$$

where x and y are independent of one another, expresses a highly characteristic property of the function f. It will, in fact, be recognized as the characteristic property of the logarithmic function; namely, that which expressed in words is the rule that "the sum of the logarithms of any two numbers is the logarithm of their product."

16. The solution of a functional equation consists in finding the most definite expression for the unknown function which will include all the functions which have the given property. When the equation contains two independent variables, like that above, the solution is effected if we can separate the variables, so that each occurs on one side only of the equation. For, by so doing, we must necessarily obtain an expression which has the same value for *any two* (and therefore for *all*) values of the independent variable.

Such an expression must have a constant value, and this fact gives us an equation containing only one variable.

For example, suppose that, with respect to an unknown function f, it can be shown that

$$xf(x) = zf(z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while x and z are independent, so that x may vary while z retains a fixed value. Let c be the value of the expression $zf(z)$ when z has a given fixed value. Then, by virtue of equation (1),

$$xf(x) = c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this equation, c is a *constant* because it is independent of x; hence it appears that, in this case, the expression $xf(x)$ is *not* itself a function of x, because it has a value independent of x. It follows that

$$f(x) = \frac{c}{x}$$

is the most definite expression we can give to $f(x)$, unless some other known property of the function serves to determine the constant c.

Examples I.

1. (α) For what value of n does x^n cease to be a function of x? (β) For what values of x does it cease to be a function of n?

(α) When $n = 0$. (β) When $x = 1$ and when $x = 0$.

2. If $y\left(1 - \dfrac{a - x}{a + x}\right) = x + \dfrac{ax - x^2}{a + x}$, show that y is a function of a, but not of x.

3. Show that $\sin x \tan \frac{1}{2}x + \cos x$ is not a function of x.

4. If $y = x + \sqrt{(1 + x^2)}$, show that $y^2 - 2xy$ is not a function of x.

5. If $f(x) = x^2$, find the value of $f(x + h)$; of $f(2x)$; of $f(x^2)$; of $f(x^2 - x)$; of $f(1)$; of $f(12)$; of $f[f(x)]$.

6. If $f(x) = \log(1 + x)$, find the value of $f(0)$; of $f(1)$; of $f(\infty)$; of $f(-1)$.

7. If $f(\theta) = \cos\theta$, find the value of $f(0)$; of $f(\frac{1}{3}\pi)$; of $f(\frac{1}{2}\pi)$; of $f(\pi)$; of $f(\frac{5}{4}\pi)$.

8. If $F(x) = a^x$, give the value of $F(a)$; of $F(1)$; of $F(0)$. Also show that in this case $[F(x)]^2 = F(2x)$.

9. Given $xy - 2x + y = n$, show that y is not a function of x when $n = 2$.

10. Given $y^2 - 2ay + x^2 = 0$, make y an explicit function of x, and draw the graph of the function. $y = a \pm \sqrt{(a^2 - x^2)}$.

11. Given $1 + \log_a y = 2 \log_a (x + a)$, make y an explicit function of x.

$$y = \frac{(x + a)^2}{a}.$$

12. Given the equations:

$$n + 1 = n(\cos^2\theta' + \cos\theta' \cos\theta + \cos^2\theta),$$

and

$$n - 1 = n(\sin^2\theta' + \sin\theta' \sin\theta + \sin^2\theta);$$

regarding θ as the independent variable, determine θ' and n as explicit functions of θ.

$$\theta' = \theta \pm \tfrac{1}{2}\pi, \text{ and } n = \mp \frac{1}{\sin\theta \cos\theta}.$$

13. Given $\sin^{-1}x - \sin^{-1}y = \alpha$, make y an explicit function of x.

$$y = x\cos\alpha \pm \sin\alpha \sqrt{(1 - x^2)}.$$

14. Given $\tan^{-1}x + \tan^{-1}y = \alpha$, make y an explicit function of x. Also show that x is the same function of y, and point out the corresponding peculiarity of the graph.

$$y = \frac{\tan\alpha - x}{1 + x\tan\alpha}.$$

15. If $y = \dfrac{2x - 1}{3x - 2}$, show that the inverse function is of the same form.

16. If $y = f(x) = \dfrac{1 + x}{1 - x}$, find $z = f(y)$, and express z as a function of x, that is to say, find $ff(x)$. Also find $ffff(x)$.

$$z = -\frac{1}{x}.$$

17. Find the inverse of the function $y = \log_a[x + \sqrt{(1 + x^2)}]$.

$$x = \tfrac{1}{2}(a^y - a^{-y}).$$

18. For what ranges of values of x is $\tan x$ a continuous function, and for what ranges an increasing function?

19. If both f and ϕ denote increasing functions, and also if both denote decreasing functions, show that $\phi[f(x)]$ is an increasing function. Show also that the inverse of an increasing runction is an increasing function.

20. Show, by consideration of the graph, that a function which is continuous for all values of x can be infinite only when x is infinite.

21. If a function continuous within a certain range is sometimes an increasing and sometimes a decreasing one, show that its inverse cannot be a one-valued function.

22. State the peculiarity of the graph of a one-valued function having the property

$$f(x) = f(-x);$$

and show that, if the function ϕ is defined by $f(x) = \phi(x^2)$, under this condition only can ϕ be a one-valued function.

23. If $f(x)$ is an unknown function having the property

$$f(x) + f(y) = f(xy),$$

prove that

$$f(1) = 0.$$

Put $y = 1$.

24. If $f(x)$ has the property

$$f(x+y) = f(x) + f(y),$$

prove that $f(0) = 0$. Also prove that the function has the property

$$f(px) = pf(x);$$

in which p is a positive or negative integer.

For positive integers, put $y = x$, $2x$, $3x$, *etc. in the given equation; for negative integers, put* $y = -x$.

25. If f denotes the same function as in Example 24, prove that

$$f(mx) = mf(x),$$

m denoting any fraction.

Put $z = \frac{p}{q}x$, *p and q being integers, and apply the result of Ex.* 24.

26. Denoting by f the same function as in the preceding examples, derive from the general property

$$f(mx) = mf(x)$$

the result

$$\frac{1}{z}f(z) = \frac{1}{x}f(x),$$

and thence deduce the form of the function. *See Art.* 16.

$$f(x) = cx.$$

27. Given $[\phi(x)]^x = [\phi(z)]^z$, and $\phi(1) = e$, determine $\phi(x)$.

$$\phi(x) = e^{\frac{1}{x}}.$$

28. Given $\phi(x) + \phi(y) = \phi(xy)$,

prove $\phi(x^m) = m\phi(x)$,

and thence prove $\phi(x) = c \log x$.

Use the methods of Examples 23, 24, *and* 25.

II.

Rates of Variation.

17. In the Differential Calculus, quantities susceptible of continuous variation are treated of by means of the *rates* of their variation. Let a continuous variable be represented, as in Art. 1, by the distance OP of a point moving along a straight line from a fixed origin of distances taken on the line; then the rate of increase of the variable is represented by the velocity of the moving point. If the line is horizontal, as in Fig. 6, distances to the right being, as usual, regarded as positive, a rate of *increase* is represented by motion to the right, and this is taken as a *positive rate.* Accordingly, motion toward the left corresponds to a rate of algebraic decrease, or a negative rate.

Denote the variable by x, and denote by t the time, as measured from some fixed instant taken as the origin of time; then, in this representation, we are in effect making x a function of t. Negative values of t correspond to positions occupied by P at instants before that chosen as the origin of time.

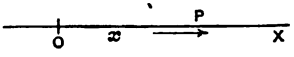

FIG. 6.

Constant Rates.

18. A variable is said to have a *constant rate,* when it receives equal *increments* in any equal intervals of time; in other words, when the differences of the values corresponding to the beginnings and ends of any equal intervals are equal. Under these circumstances, the point P in the graphic illustration, Fig. 6, will have a uniform velocity. The increments of the variable x mentioned above are now represented by spaces described by P in intervals of time having some fixed magnitude. The definition of uniform motion or *constant velocity* requires that these spaces should be equal. It readily follows that *the spaces passed over in any intervals of time are proportional to the intervals,* and this may be taken as a more convenient definition of a constant velocity.

19. The numerical measure adopted for a constant velocity is the number of units of space passed over in the unit of time (generally the second), and accordingly the measure of a constant rate for any variable is the increment received in a unit of time. Let k denote this increment (or, in the illustration, the space described in a unit of time), then the increment received in t units of time by a variable x having the constant rate k will be kt. Hence, if a denotes the value of x at the time $t = 0$, the value of x at the end of t units of time will be

$$x = a + kt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus a uniformly varying quantity is a *linear function of the time* elapsed since a given instant, that is, of the time as measured from a fixed origin of time.

Conversely, whenever a variable x is a linear function of the time, it has a uniform rate; and the coefficient of t, when x is expressed in the form (1), is the measure of this rate.

Variable Velocities.

20. If the velocity of a point be *not* uniform, *its numerical measure at any instant is the number of units of space which would have been described in a unit of time, if the velocity had remained constant from and after the given instant.*

Thus, when we speak of a body as having at a given instant a velocity of 32 feet per second, we mean that, should the body continue to move during the whole of the next second with the same velocity which it had at the given instant, 32 feet would be described. The *actual* space described may be greater or less, in consequence of the change in velocity which takes place during the second; it is, for instance, in the case of a falling body greater than the measure of the velocity at the beginning of the second, because the velocity increases throughout the second.

21. Attwood's machine for determining experimentally the velocities acquired by falling bodies furnishes an example of the practical application of the principle embodied in the above definition.

This apparatus consists essentially of a thread passing over a fixed pulley, and sustaining equal weights one at each extremity, the pulley being so constructed as to offer the slightest possible resistance to turning. On one of the weights a small bar of metal is placed, which, destroying the equilibrium, causes the weight to descend with an increasing velocity. To determine the value of this velocity at any point, a ring is so placed as to intercept the bar at that point, and allow the weight to pass. Thus, the sole cause of the variation of the velocity having been removed, the weight moves on uniformly with the required velocity, and the space described during the next second becomes the measure of this velocity.

Variables with Rates not Uniform.

22. When a variable quantity x is represented, as in Fig. 6, by a distance measured upon a fixed straight line, the increment which it receives in a certain interval of time is represented by the space passed over by the moving point in that interval. Such an increment (or difference of values of x) is denoted by the symbol Δx, and the interval of time to which it corresponds is denoted by Δt. Then the characteristic of a variable which has not a constant rate is that Δx is *not* proportional to Δt, so that neither the increment received in a unit of time, nor the ratio of any simultaneous increments, furnishes a measure of the rate. Under these circumstances, the measure of the rate at a given instant is taken to be the same as that of the variable velocity which represents it; namely, it is ***the increment which would have been received in a unit of time, if the rate had remained constant for the whole of that interval.***

Differentials.

23. The increment which would be received by x in any chosen interval of time, on the hypothesis made above, is called a *differential.* Such an hypothetical increment is denoted by dx; while the interval of time to which it corresponds is denoted by dt and is called *the differential of time.* The differential of time can be chosen of any magnitude; and, considering several values of dt, the corresponding values of dx will be proportional to them. It follows that the ratio

$$\frac{dx}{dt}$$

will be the measure of the rate of x, no matter what the value of dt.

24. If now we put $dt = \Delta t$ we shall *not,* in general, have $dx = \Delta x$. This will be true only when x varies with a constant rate; that is to say, when x is a linear function of t. Therefore, writing the equation

$$\frac{\Delta x}{\Delta t} = \frac{dx}{dt} + e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

e is a quantity which in general will have a positive or negative value.

This quantity e is the difference between the rate of x and the ratio of certain actual corresponding increments of x and t. Since this difference is due solely to the fact that the rate of x changes during the interval Δt, e is a function of Δt which *vanishes with* Δt.

25. With this understanding of the nature of e, equation (1) expresses the fact that *the limiting value of the ratio of the* *increments Δx and Δt, when Δt is diminished without limit, is equal to the measure of the rate of x.*[*]

Formula for the Non-Linear Function of t.

26. A variable x which has not a uniform rate is a *non-linear* function of t. Such a function may be expressed in a form which exhibits the difference between it and the linear function. For this purpose, we make use of the value which x has at any particular instant, and for simplicity take that instant as the origin of time. Denote by a the special value of x thus taken to correspond to $t = 0$; this may be called *the initial value of x*. Then, passing to any other corresponding values of x and t, we have

$$t = \Delta t \quad \text{and} \quad x = a + \Delta x. \quad . \quad . \quad . \quad . \quad (2)$$

Denote also the value which the rate of x has when $t = 0$, that is, *the initial value of the rate*, by k. Now, since the rate is not constant, we have, by equation (1), Art. 24,

$$\Delta x = (k + e)\,\Delta t;$$

whence, equation (2) becomes

$$x = a + (k + e)\,t, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where e is, by Art. 24, a function which vanishes when $t = 0$.

27. It follows that, when a given function of t is put in this form, k is the value which the coefficient of t assumes when we put $t = 0$. Thus, if the coefficient of t is variable,

---

[*] In accordance with this theorem, the hypothetical differences or differentials, corresponding to any particular simultaneous values of x and t, are quantities which (while they may have any magnitudes whatever) must always have that ratio which is the limiting value of the ratios of the actual increments, when made indefinitely small.

the rate is variable, and its value when $t = 0$ is obtained by putting $t = 0$ in the coefficient.

We have seen in Art. 23 that $\frac{dx}{dt}$ is the symbol for the rate of x. When this is variable, the value of it which corresponds to a special value a of t is denoted by $\left.\frac{dx}{dt}\right]_a$. Thus the theorem just proved is that, when x is put in the form

$$x = a + (k + e)t,$$

where e vanishes with t,

$$\left.\frac{dx}{dt}\right]_a = k.$$

The Differential of the Sum of Several Variables.

28. Let x and y denote any two variables and k and k' their rates at any given instant. Then, taking this instant as the origin of time, their values may, by the preceding articles, be written

$$x = a + (k + e)t, \quad \dots \dots \quad (1)$$

$$y = b + (k' + e')t. \quad \dots \dots \quad (2)$$

Hence their sum is

$$x + y = (a + b) + (k + k' + e + e')t, \quad \dots \quad (3)$$

which is of the same form. Since $e + e'$ vanishes when $t = 0$, the coefficient of t takes the value $k + k'$ when $t = 0$. Thus, by Art. 27,

$$\left.\frac{d(x + y)}{dt}\right]_0 = k + k' = \left.\frac{dx}{dt}\right]_0 + \left.\frac{dy}{dt}\right]_0.$$

The several rates in this equation are their values at the instant chosen as the origin of time. But, since any instant may be so chosen, we have proved that at all times

$$\frac{d(x+y)}{dt}=\frac{dx}{dt}+\frac{dy}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, the rate of the sum of two variables is always equal to the sum of their rates.

29. Multiplying by dt we have

$$d(x+y)=dx+dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which it is of course to be understood that all the differentials correspond to the same value of dt.

This formula is readily extended to any number of variables. Thus

$$\begin{aligned} d(x+y+z+\ldots) &= dx+d(y+z+\ldots) \\ &= dx+dy+dz+\ldots \quad . \quad . \quad . \, (A) \end{aligned}$$

that is, *the differential of the sum of any number of variables is the sum of their differentials.* Since a constant has no differential it appears that in differentiating a polynomial, constant terms do not affect the result.

The Differential of a Term having a Constant Coefficient.

30. Let the term be denoted by mx, m denoting a constant. Putting, as in Art. 26,

$$x=a+(k+e)t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we have

$$mx=ma+(mk+me)t. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since me vanishes when $t = 0$, the coefficient of t assumes the value mk, and we have, by Art. 27,

$$\left.\frac{d(mx)}{dt}\right]_0 = mk = m\left.\frac{dx}{dt}\right]_0 .$$

Hence, at the instant chosen as the origin of time, the rate of mx is m times the rate of x. But, as this may be any instant, the same thing is true generally, and multiplying by dt,

$$d(mx) = mdx.$$

It therefore follows that *the differential of a term having a constant coefficient is equal to the product of the differential of the variable factor by the constant coefficient.*

The constant m may have a negative value, and in particular

$$d(-x) = -dx.$$

The Differential of the Product of Two Variables.

31. Let x and y be any two variables, and, selecting any instant as the origin of time, express them, as in Art. 26, by

$$x = a + (k + e)t,$$
$$y = b + (k' + e')t.$$

Then their product is

$$xy = ab + [bk + ak' + be + ae' + (k + e)(k' + e')t]t.$$

Assuming the initial values and rates, a, b, k and k', to be finite quantities, the terms be and ae' as well as $(k + e)(k' + e')t$ vanish with t; hence the coefficient of t reduces when $t = 0$ to $bk + ak'$, which is therefore, by Art. 27, the measure of the rate at the instant $t = 0$. Thus, at the chosen instant, the rate of xy is the sum of the products of the values of each variable and the rate of the other.

Since this is true at every instant, we have therefore in general

$$\frac{d(xy)}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt};$$

whence

$$d(xy) = xdy + ydx.$$

Examples II.

1. Find the differential of $\frac{2x}{3a}$ and of $\frac{x}{m-2}$.

$$\frac{2dx}{3a} \text{ and } \frac{dx}{m-2}.$$

2. Find the differential of $\frac{x+a}{m^2}$ and of $\frac{a-x}{m^2}$.

$$\frac{dx}{m^2} \text{ and } -\frac{dx}{m^2}.$$

3. Find the differential of $\frac{a+b+(a-b)x}{a^2-b^2}$.

$$\frac{dx}{a+b}.$$

4. Find the differential of $\frac{a+x}{a+b}$ and of $\frac{b(x+y)}{a(a+b)}$.

$$\frac{dx}{a+b} \text{ and } \frac{b(dx+dy)}{a(a+b)}.$$

5. Given $ay + bx + 2cx + ab = 0$, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{b+2c}{a}.$$

6. Given $y \log a + x \sin \alpha - y \cos \alpha - ax + \tan \alpha = 0$, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{a - \sin \alpha}{\log a - \cos \alpha}.$$

7. Given $ay \cos^2 \alpha - 2b(1 - \sin \alpha)x = b(a - x \cos^2 \alpha)$, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{b(1 - \sin \alpha)}{a(1 + \sin \alpha)}.$$

8. Given $a^2 + 2(1 + \cos \alpha)y = (x + y) \sin^2 \alpha$, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \tan^2 \frac{\alpha}{2}.$$

9. Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, to express dz in terms of dx and dy.

$$dz = -\frac{c}{a}\, dx - \frac{c}{b}\, dy.$$

10. The sides of a rectangle have the values 12 and 9 inches at a given instant, they are then increasing uniformly, the first at the rate of 2 inches per second and the second at the rate of 1 inch per second. At what rate per second is the area increasing?

30 square inches.

11. How would the area be changing if the first side were decreasing, other things being as in Ex. 10?

12. In each of the last two examples what will be the rate after the lapse of one second?

13. A man whose height is 6 feet walks directly away from a lamp-post at the rate of 3 miles an hour. At what rate is the extremity of his shadow travelling, supposing the light to be 10 feet above the level pavement on which he is walking?

Draw a figure, and denote the variable distance of the man from the lamp-post by x, and the distance of the extremity of his shadow from the post by y. $7\frac{1}{2}$ miles per hour.

14. At what rate does the man's shadow (Ex. 13) increase in length?

III.

Rate of a Function of an Independent Variable.

32. In many applications of the Calculus, the variables treated of are actual functions of the time, and therefore have definite rates, which being generally variable are themselves functions of the time. The spaces passed over by the falling body in Arts. 20 and 21 afford an illustration: the velocities as well as the spaces are definite functions of the time.

In other applications, the variables concerned, although connected together, have no necessary connection with elapsing time. In these cases, one of the variables is arbitrarily chosen as the independent variable. Denoting it by x, it is assumed to have a rate $\frac{dx}{dt}$ of arbitrary and constant value. Then, if y is a given function of x, its rate $\frac{dy}{dt}$ will depend for its value not only upon the functional relation of y to x, but also upon the rate assumed for x.

33. Now, since dy and dx in the symbols for the rates imply the same value of dt, we have

$$\frac{\text{rate of } y}{\text{rate of } x} = \frac{dy}{dx},$$

that is to say, the ratio of the differentials expresses *the relative rate of y* when the rate of the independent variable x is taken as the standard. Since the rate of x is assumed to be constant, the relative rate of y will, in general, be variable; that is, it will be a function of the independent variable.

But it is important to notice that this ratio $\frac{dy}{dx}$ is *not* a function of dx; for the value of it which corresponds to a special value of x is absolutely determined by the rates of y and x, and yet the values of dy and dx can be changed by altering the value of dt.

The Derivative.

34. It follows that, when

$$y = f(x), \quad \dots\dots\dots \quad (1)$$

we may put

$$\frac{dy}{dx} = f'(x), \quad \dots\dots\dots \quad (2)$$

where f' is a new function of x. The new function thus derived from the given function f is called *the derived function*, or more commonly *the derivative* of the function f.

Equation (2) may also be written in the form

$$dy = f'(x)dx, \quad \dots\dots\dots \quad (3)$$

and, for this reason, the derivative $f'(x)$ is also sometimes called *the differential coefficient of y*.

It is only in the case of the linear function

$$y = mx + b,$$

which gives

$$dy = mdx,$$

that the differential coefficient is a constant and not a function of x.

35. We are said *to differentiate* a function of a single independent variable, when we express its differential in the form (3). The first part of our work in the following chapters will be to obtain the formulæ or rules for the differentiation of the simple functions in this form. The expression of the value of $\frac{dy}{dx}$ in the form (2) is called *taking the derivative of y with respect to x.*

It is to be noticed that the result of this operation, of which the symbol is $\frac{d}{dx}$, is simpler than that of differentiation, of which the symbol is d. The reason is that dy is a function of the independent quantity dx[*] as well as of x; whereas the dx implied in the value of dy, equation (3), has been removed from the derivative by division.

Graphic Representation of the Derivative.

36. When the graph of the function $y = f(x)$ is drawn, as in Art. 6, the simultaneous values of x and y are the rectangular coordinates of a moving point which describes a continuous line. If the function is linear, this line is straight, and the moving point is said to have a *constant direction.* We have seen in Art. 34 that, in this case, the ratio of the rate of y to that of x is constant and equal to m, which, in the equation $y = mx + b$, is the tangent of the angle which the straight line makes with the axis of x.

37. For all other functions, a curve is described and the direction of the moving point is variable. Let the curve in Fig. 7 be the graph of the function $y = f(x)$. Since we have assumed (Art. 32) the rate of x to be constant, the rate of y is now variable. When the moving point arrives at a particular position, as P in the diagram, let us suppose the rate of y to become constant without suffering change in value, The moving point will then

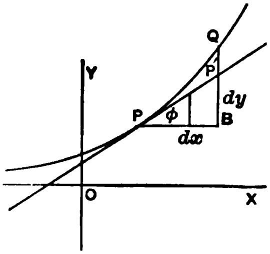

FIG. 7.

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[*] Thus d is not a functional symbol, for dx is independent of x. We cannot however say that dy is independent of y, since it is a function of x upon which y depends, and therefore has different values for different values of y.

describe a straight line PP', and the constant direction of this line exhibits the direction of the moving point in the curve at the instant it passes the point P.

This straight line passing through a given point of a curve and having the direction of the curve at that point is called the *tangent* to the curve at that point, which is called the point of contact.

38. Let PP', Fig. 7, be the space described in the time dt by the point considered in the preceding article, then the difference of abscissæ PB represents dx, and the difference of ordinates BP' represents dy. The fact that the ratio of these differentials is independent of their magnitude is illustrated by the similar triangles in the figure.

Let ϕ denote the inclination of the tangent line to the axis of x, then we have

$$\frac{dy}{dx} = \frac{P'B}{PB} = \tan\phi.$$

Thus the trigonometric tangent of the inclination of the graph of $y = f(x)$ is the graphic representation of the derivative of the function.

It will be noticed that there are two values of ϕ, differing by 180°, according as we suppose the point to move in one or the other of the two opposite directions in the curve; but the value of $\tan\phi$ is the same for these two values, since $\tan(\phi + 180°) = \tan\phi$.

Sign of the Derivative.

39. By the definition given in Art. 11, a function $y = f(x)$ is an increasing one, when x and y increase together or decrease together; in other words, when the rates of x and y have the same algebraic sign. In this case, the ratio $\dfrac{dy}{dx}$

is positive. Thus $f(x)$ is an increasing function for all values of x which make the derivative $f'(x)$ positive; and, on the other hand, it is a decreasing function for all values which make $f'(x)$ negative. Accordingly, when the graph is drawn, $\tan\phi$ is taken as the *gradient* or measure of the variable slope of the curve which, as stated in Art. 12, is positive in the first case and negative in the second.

Limit of the Ratio of Differences.

40. Denoting actual increments, as in Art. 24, by Δt, Δx and Δy, we now have $\Delta x = dx$ when $\Delta t = dt$, because x is assumed to have a uniform rate. But we have *not* $\Delta y = dy$. The difference is illustrated in Fig. 7, where the actual increment of the ordinate is $\Delta y = BQ$, terminating in the curve, while the hypothetical increment is $dy = BP'$, terminating in the tangent line. If, after the analogy of Art. 24, we put

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

e will be a quantity which vanishes with Δx. Thus when $y = f(x)$, *the derivative is the limiting value of the ratio of simultaneous increments or differences, when the absolute values of these increments are diminished without limit.**

41. The quantity e in equation (1) is illustrated in Fig. 7 by the ratio of $P'Q$ to PB. Not only does $P'Q$ vanish with PB or Δx, but its ratio to Δx vanishes. The first member of the equation is the trigonometric tangent of the inclination, that is to say, the slope of a secant line which cuts the graph in the points P and Q, of which the abscissæ are x and $x + \Delta x$. The ratio in the second member is the slope of the tangent at the former point. Thus the geometrical equivalent of the proposition stated above is that *the tangent is the limiting position of the secant line.*

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* The actual increments or differences are sometimes called *finite differences* in distinction from the small differences of which the limits only are considered in the Differential Calculus. The latter are then called infinitesimal differences, an *infinitesimal* being defined as a quantity having zero for its limit.

The special value which the derivative $\frac{dy}{dx}$ takes when x has a special value a is denoted by $\left.\frac{dy}{dx}\right]_a$. Thus, if $y = f(x)$,

$$\frac{dy}{dx} = f'(x) \qquad \text{and} \qquad \left.\frac{dy}{dx}\right]_a = f'(a).$$

Examples III.

1. If a point moves in the straight line $2y - 7x - 5 = 0$, so that its ordinate decreases at the rate of 3 units per second, at what rate is the point moving in the direction of the axis of x?

$$\frac{dx}{dt} = -\frac{6}{7}.$$

2. If a point starting from $(0, b)$ moves so that the rates of its co-ordinates are k and k', show that its path is $y = mx + b$, m being equal to $\frac{k'}{k}$.

Express x and y in terms of t (Art. 19) and eliminate t.

3. If a point moving in a curve passes through the point $(5, 3)$ moving at equal rates upward and toward the left, find the value of $\left.\frac{dy}{dx}\right]_5$, also the equation of the tangent line to the curve at the given point.

$$\left.\frac{dy}{dx}\right]_5 = -1; y + x = 8.$$

4. If a point is moving in the straight line

$$x \cos \alpha + y \sin \alpha = p,$$

its rate in the positive direction of the axis of x being $l \sin \alpha$, what is its rate of motion in the direction of the axis of y?

$$-l \cos \alpha.$$

5. Given $ay \sin \alpha - ax + ax \cos \alpha - b^2 \sec \alpha = 0$; show that ϕ is constant and equal to $\frac{1}{2}\alpha$.

6. If $f(x) = \tan x$, show that $f'(x)$ must always be positive.

7. Show, by tracing the graph, that if $y = x^3$ $\dfrac{dy}{dx}$ can never be negative.

8. Given the property of the parabola, (which is the graph of $y = \sqrt{x}$, Fig. 2), that the subtangent is bisected at the vertex; deduce the value of the derivative.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

9. The graph of the function $y = \sqrt{(a^2 - x^2)}$ is the circle $x^2 + y^2 = a^2$. Deduce the value of the derivative from the property that the tangent is perpendicular to the radius.

$$\frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}}.$$

CHAPTER II.

FORMULÆ AND METHODS OF DIFFERENTIATION.

IV.

Character of the required Formulæ.

42. THE functions of an independent variable are expressed by means of a few simple functional symbols and their combinations, either by algebraic operations, or in the form of a function of a function. It is the object of the present Chapter to establish the formulæ for the differentials of the simple functions of one independent variable, and also the formulæ by means of which we can apply these to the differentiation of the more complex functions formed by combining the elementary forms.

43. Of the latter class of formulæ we have already found those for the sum and for the product of two variables, Arts. 29 and 31. From the latter we shall see, in the next section, that all the formulæ for algebraic functions may be derived. We shall, however, first give a method of deducing the derivative of the square by means of a functional equation, and shall derive from it another proof of the formula for the product.

Differentiation of the Square.

44. Let $f(x) = x^2$; it is required to find $f'(x)$. Assume another value of the independent variable connected with x by the relation

$$z = mx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where m is a constant. We propose to find a relation between the corresponding values of the derivative.

Since the members of equation (1) remain equal while each is variable, their rates and consequently their differentials are equal. In other words, we can *differentiate the equation.* Thus, by Art. 30,

$$dz = m dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In like manner, from the relation between the functions, which is

$$z^2 = m^2 x^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we have, by differentiation,

$$d(z^2) = m^2 d(x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Dividing equation (4) by equation (2), we have

$$\frac{d(z^2)}{dz} = m \frac{d(x^2)}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

a relation between the derivatives $f'(z)$ and $f'(x)$; that is, between the values of the derivative of the function "square" which correspond to two values of the independent variable connected by equation (1). These values are thus not independent of one another in equation (5); but, if we eliminate m by means of equation (1), thus obtaining

$$\frac{1}{z}\frac{d(z^2)}{dz} = \frac{1}{x}\frac{d(x^2)}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

$$\frac{1}{z} f'(z) = \frac{1}{x} f'(x),$$

we have a relation in which x and z are entirely independent, because we obtain the same result no matter what the value of m. In this equation the variables are already separated, as in Art. 16; accordingly, the members are of the same form and constitute an expression which does not vary with x. Hence, denoting its constant value by c, we have

$$\frac{1}{x}f'(x) = c\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

therefore

$$f'(x) = \frac{d(x^2)}{dx} = cx,^* \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

or

$$d(x^2) = cxdx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Applying this to the identity

$$(x + h)^2 = x^2 + 2hx + h^2,$$

where h is a constant, we have

$$c(x + h)dx = cxdx + 2hdx\,;$$

whence $c = 2$,† and equation (9) becomes

$$d(x^2) = 2xdx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

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* Referring to the graph of this function, Fig. 1, p. 4, it will be seen that the meaning of the process is that, if two points move on the curve in such a manner that their abscissæ have the constant ratio m, the rates of the abscissæ have the ratio m, and those of the ordinates the ratio m^2. Hence the slopes or values of tan ϕ, see equation (5), have the ratio m, that is, the same ratio as the abscissæ. That is to say, *the slope is proportional to the abscissa.*

† Had we commenced with the relation $z = x + h$ we should have obtained

$$\frac{d(z^2)}{dz} - 2z = \frac{d(x^2)}{dx} - 2x = C,$$

in which the constant is easily shown to be zero.

The Differential of the Product.

45. If we apply the formula just derived for the square to the identity

$$(x+y)^2 = x^2 + 2xy + y^2,$$

regarding the differential of the product xy to be as yet unknown, we find

$$2(x+y)(dx+dy) = 2xdx + 2d(xy) + 2ydy,$$

which reduces to

$$d(xy) = xdy + ydx.$$

This formula is readily extended to products consisting of any number of factors. Thus, let $x_1x_2x_3 \ldots . x_p$ denote the product of p variable factors, then

$$\begin{aligned} d(x_1x_2x_3 \ldots . x_p) &= x_2x_3 \ldots x_pdx_1 + x_1d(x_2x_3 \ldots x_p) \\ &= x_2x_3 \ldots x_pdx_1 + x_1x_3 \ldots x_pdx_2 + x_1x_2d(x_3 \ldots x_p) \\ &= x_2x_3 \ldots x_pdx_1 + x_1x_3 \ldots x_pdx_2 \ldots + x_1 \ldots x_{p-1}dx_p. \qquad \text{(B)} \end{aligned}$$

Hence *the differential of the product of several variables is the sum of the products of the differentials of the factors each multiplied by all the other factors.* In other words, it is the sum of the differentials obtained by supposing each factor in turn to be the only variable one.

Differentiation of an Inverse Function.

46. When the derivative of a function is known, the derivative of the function inverse to it is readily deduced. Thus, in the case of the square root, which is inverse to the square, let

$$y = \sqrt{x}, \qquad \text{whence} \qquad x = y^2.$$

Taking the derivative of this function of y, we have

$$\frac{dx}{dy} = 2y, \qquad \text{whence} \qquad \frac{dy}{dx} = \frac{1}{2y}.$$

Finally, expressing this inverse derivative in terms of x,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \qquad \text{or} \qquad dy = \frac{dx}{2\sqrt{x}};$$

that is,

$$d\sqrt{x} = \frac{dx}{2\sqrt{x}}.$$

Differentiation of a Function of a Function.

47. Since the ratio of differentials is independent of their actual values, x in the formula just obtained may be replaced by a variable which is not independent, and may not have a constant rate. Accordingly, the formula, when expressed as a rule, becomes: *The differential of the square root of any variable is the result of dividing the differential of the variable by twice the given square root.* For example, if $a^2 + x^2$ is the variable which takes the place of x in the formula, we have

$$d\sqrt{(a^2 + x^2)} = \frac{d(a^2 + x^2)}{2\sqrt{(a^2 + x^2)}} = \frac{2x dx}{2\sqrt{(a^2 + x^2)}} = \frac{x}{\sqrt{(a^2 + x^2)}} dx.$$

48. So in general, for any "function of a function," or expression of the form

$$y = \phi[f(x)],$$

if the rules for the differentiation of the functions ϕ and f are known, we can differentiate y, that is, express dy in terms of x and dx. For, if $z = f(x)$, we have $y = \phi(z)$, whence

$$dy = \phi'(z)dz = \phi'(z)f'(x)dx,$$

in which the functions ϕ' and f' are supposed known.

It is for this reason that, as mentioned in Art. 43, it is only necessary to prepare formulæ for the differentiation of the simple functions.

The Rates of Geometrical Variables.

49. Geometrical magnitudes dependent upon the position of a point become variables having definite rates, when the velocity and direction of the point are given. We have already employed the velocity of the moving point to represent the rate of its distance from a fixed point in the line of its motion supposed straight. Such a distance, when it occurs in a problem, is thus marked out as the most convenient independent variable x, in terms of which to express any other variable y of which the rate is required. For, the rate of x will be known, and the derivative with respect to x is the relative rate of y.

50. As an illustration, suppose a man to be walking on a straight path BC at the rate of 5 feet per second: required the rate of change in his distance AP from a point A at the perpendicular distance $AB = 120$ feet from the path, at the instant when he is passing the point C, 50 feet from the foot of the perpendicular. Denote the variable distance of the man from B by x, so that $\frac{dx}{dt}$ may denote the known velocity of P, and denote the constant AB by a. Then, by geometry, the variable distance of which the rate is required is

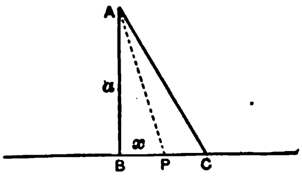

FIG. 8.

$$y = \sqrt{(a^2 + x^2)}.$$

Hence its rate is

$$\frac{dy}{dt}=\frac{d\sqrt{(a^2+x^2)}}{dt}=\frac{x}{\sqrt{(a^2+x^2)}}\cdot\frac{dx}{dt}.$$

Substituting herein the special value $x = 5$, together with $a = 120$ and $\frac{dx}{dt} = 5$, we find $\frac{dy}{dt} = 1\frac{12}{13}$, which is therefore the rate at which the distance AP is increasing at the instant P passes C.

51. We might, in the solution of this problem, have expressed y directly in terms of t. For this purpose, assume the instant when P passes B as the origin of time. Then the value of BP at the end of the time t is $5t$, and that of AP is

$$y=\sqrt{(120^2+25t^2)}=5\sqrt{(24^2+t^2)},$$

whence

$$\frac{dy}{dt}=\frac{5t}{\sqrt{(576+t^2)}}.$$

The man is at C when $t = 10$; therefore the rate required is

$$\left.\frac{dy}{dt}\right]_{10}=\frac{50}{26}=1\frac{12}{13},$$

as before.

Again, we have

$$\left.\frac{dy}{dt}\right]_0=0, \quad \text{and} \quad \left.\frac{dy}{dt}\right]_\infty=5.$$

Of these results, the first shows that when P is at B it is neither approaching nor receding from A; and the second shows that the rate of receding from A has for its limiting value the actual velocity of P, or rate at which it recedes from B.

Examples IV.

1. Differentiate $(2x + 3)^2$, and find the numerical value of its rate, when x has the value 8, and is decreasing at the rate of 2 units per second.

The differential required is denoted by $d\,[(2x + 3)^2]$, *and the rate by* $\frac{d[(2x + 3)^2]}{dt}$; *the given rate* $\frac{dx}{dt} = -2$. -152 units per second.

2. Find the numerical value of the rate of $(x^2 - 2x)^2$, when $x = 3$ and is increasing at the rate of $\frac{1}{2}$ of one unit per second.

Differentiate the given expression before substituting.

12 units per second.

3. Find the numerical value of the rate of $\sqrt{(y^2 + x^2)}$, when $y = 7$ and $x = -7$, if y is increasing at the rate of 12 units per second, and x at the rate of 4 units per second.

$4\sqrt{2}$ units per second.

4. If $f(x) = x - \sqrt{(x^2 - a^2)}$, find $f'(x)$, and show that $f(x)$ is a decreasing function.

$$f'(x) = 1 - \frac{x}{\sqrt{(x^2 - a^2)}}.$$

5. Differentiate the identity $(\sqrt{x} + \sqrt{a})^2 = x + a + 2\sqrt{ax}$, and show that the result is an identity.

6. Differentiate $\sqrt{\left(\frac{x^2 - 2ax}{a^2 - 2ab}\right)}$.

The constant factor $\frac{1}{\sqrt{(a^2 - 2ab)}}$ *should be separated from the variable factor before differentiation.*

$$\frac{1}{\sqrt{(a^2 - 2ab)}} \cdot \frac{x - a}{\sqrt{(x^2 - 2ax)}}\,dx.$$

7. If $f(x) = (1 + x^2)^{\frac{1}{2}}$, $\qquad f'(x) = \frac{x}{(1 + x^2)^{\frac{1}{2}}}$.

8. If $f(x) = \sqrt{(a^3 + 2b^2x + cx^2)}$,

$$f'(x) = \frac{b^2 + cx}{\sqrt{(a^3 + 2b^2x + cx^2)}}.$$

9. If $f(x) = \sqrt{[x + \sqrt{(1 + x^2)}]}$, $\qquad f'(x) = \frac{\sqrt{[x + \sqrt{(1 + x^2)}]}}{2\sqrt{(1 + x^2)}}$.

10. If $f(x) = \frac{a^2}{x - \sqrt{(x^2 - a^2)}}$, $\qquad f'(x) = 1 + \frac{x}{\sqrt{(x^2 - a^2)}}$.

Rationalize the denominator before differentiating.

11. Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, express $\frac{dy}{dx}$ in terms of x, and give the values of $\left.\frac{dy}{dx}\right]_0$ and $\left.\frac{dy}{dx}\right]_a$. $\qquad \frac{dy}{dx} = \mp \frac{b}{a} \cdot \frac{x}{\sqrt{(a^2 - x^2)}}$.

12. Given $y^2 = 4ax$, express $\frac{dy}{dx}$ in terms of x, also in terms of y, and give the values of $\left.\frac{dy}{dx}\right]_a$ and $\left.\frac{dy}{dx}\right]_{4a}$. $\qquad \frac{dy}{dx} = \sqrt{\frac{a}{x}} = \frac{2a}{y}$.

13. If the side of an equilateral triangle increases uniformly at the rate of 3 ft. per second, at what rate per second is the area increasing, when the side is 10 ft.? $\qquad 15\sqrt{3}$ sq. ft.

14. A stone dropped into still water produces a series of continually enlarging concentric circles; it is required to find the rate per second at which the area of one of them is enlarging, when its diameter is 12 inches, supposing the wave to be then receding from the centre at the rate of 3 inches per second. $\qquad 36\,\pi$ sq. inches.

15. If a circular disk of metal expands by heat so that the area, A of each of its faces increases at the rate of 0.01 sq. in. per second, at what rate per second is its diameter increasing?

$$\frac{1}{100\sqrt{(\pi A)}} \text{ in.}$$

16. A man standing on the edge of a wharf is hauling in a rope attached to a boat at the rate of 4 ft. per second. The man's hands being 9 ft. above the point of attachment of the rope, how fast is the boat approaching the wharf when she is at a distance of 12 ft. from it? $\qquad$ 5 ft. per second.

17. A ladder 25 ft. long reclines against a wall; a man begins to pull the lower extremity, which is 7 ft. distant from the bottom of the wall, along the ground at the rate of 2 ft. per second; at what rate per second does the other extremity *begin* to descend along the face of the wall? $\qquad$ 7 inches.

18. One end of a ball of thread is fastened to the top of a pole 35

ft. high; a man holding the ball 5 ft. above the ground moves uniformly from the bottom at the rate of five miles per hour, allowing the thread to unwind as he advances. What is the man's distance from the pole when the thread is unwinding at the rate of one mile per hour? $\frac{5}{8}\sqrt{6}$ ft.

19. A vessel sailing due south at the uniform rate of 8 miles per hour is 20 miles north of a vessel sailing due east at the rate of 10 miles per hour. At what rate are they separating—(α) at the end of $1\frac{1}{2}$ hours? (β) at the end of $2\frac{1}{2}$ hours?

Express the distances in terms of the time. (α) $5\frac{1}{17}$ miles per hour.

20. When are the two ships mentioned in the preceding example neither receding from nor approaching each other?

When $t = \frac{40}{41}$ of an hour.

V.

The Differential of the Reciprocal.

52. The differential of the reciprocal is readily obtained by means of the implicit form of this function.

Denoting the function by y, we have

$$y = \frac{1}{x}, \quad \therefore \quad xy = 1.$$

Differentiating the latter equation by the formula for the product, we obtain

$$ydx + xdy = 0,$$

whence

$$dy = -\frac{ydx}{x};$$

substituting the value of y,

$$d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}.$$

That is, *the differential of the reciprocal of a variable is the negative of the result of dividing the differential by the square of the variable.*

This formula enables us to differentiate any fraction of which the numerator is constant and the denominator a variable whose differential is known. Thus if

$$y = \frac{a^2 - b^2}{a^2 - x^2},$$

we have

$$dy = (a^2 - b^2)\, d\frac{1}{a^2 - x^2} = (a^2 - b^2)\frac{2x dx}{(a^2 - x^2)^2}.$$

Differential of the Quotient of Two Variables.

53. Since a fraction of which both terms are variable is the product of its numerator and the reciprocal of its denominator, we can now express the differential of such a fraction in terms of those of its numerator and denominator. Thus

$$d\left(\frac{x}{y}\right) = d\left(x\frac{1}{y}\right) = \frac{1}{y}dx + xd\left(\frac{1}{y}\right)$$

$$= \frac{dx}{y} - \frac{xdy}{y^2} = \frac{ydx - xdy}{y^2}. \quad . \quad . \quad . \quad (C)$$

That is to say, *the differential of a fraction is the result of taking the product of the denominator into the differential of the numerator minus that of the numerator into the differential of the denominator, and dividing by the square of the denominator.*

The signs of the terms in $ydx - xdy$ can be recollected by recalling the fact that a fraction is an increasing function of its numerator and a decreasing function of its denominator.

As an illustration of the application of this formula, we have

$$d\left(\frac{2x-a}{x^2+b}\right) = \frac{2(x^2+b) - 2x(2x-a)}{(x^2+b)^2}\,dx = 2\,\frac{b+ax-x^2}{(x^2+b)^2}\,dx.$$

Differentiation of x^n.

54. To obtain the differential of the power when the exponent is a positive integer, suppose each of the variables $x_1, x_2, \dots x_p$ in formula (B), Art. 45, to be replaced by x. The first member contains p factors, and the second p terms; the equation therefore reduces to

$$d(x^p) = px^{p-1}dx. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Next, when the exponent is a positive fraction, let $n = \frac{p}{q}$, where p and q are integers. Put

$$y = x^n = x^{\frac{p}{q}}, \qquad \text{whence} \qquad y^q = x^p.$$

This last equation can be differentiated by equation (1), because p and q are both integers. Thus

$$qy^{q-1}dy = px^{p-1}dx,$$

whence

$$dy = \frac{p}{q}\cdot\frac{x^{p-1}}{y^{q-1}}\,dx.$$

Substituting the value of y,

$$d(x^{\frac{p}{q}}) = \frac{p}{q}\cdot\frac{x^{p-1}}{x^{p-\frac{p}{q}}}dx = \frac{p}{q}x^{\frac{p}{q}-1}dx. \quad . \quad . \quad . \quad (2)$$

Finally, when the exponent is negative, we have

$$x^{-m} = \frac{1}{x^m}.$$

Differentiating by the formula of Art. 52, we obtain

$$d(x^{-m}) = -\frac{d(x^m)}{x^{2m}},$$

and, since m is a positive integer or fraction, we have, by (1) or (2),

$$d(x^{-m}) = -\frac{mx^{m-1}dx}{x^{2m}} = -mx^{-m-1}dx. \quad . \quad . \quad (3)$$

Equations (1), (2) and (3) show that, for all values of n,

$$d(x^n) = nx^{n-1}dx. \quad . \quad . \quad . \quad . \quad . \quad (a)^*$$

55. Formula (a) includes those already found for the square, the square-root, Art. 46, and the reciprocal, Art. 52, which are the special cases corresponding to $n = 2$, $n = \frac{1}{2}$ and $n = -1$. The two last-mentioned formulæ are, however, particularly useful because applicable to a familiar form of notation different from that of fractional and negative exponents.

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* The formulæ of this series are recapitulated on page 79. Together with formulæ (A), (B) and (C), for the algebraic combinations of functions, they form the body of rules for calculating differentials which properly constitute the Differential Calculus.

On the other hand, it is often useful to transform an expression, by the use of fractional and negative exponents, in order to employ the general formula (a) instead of a combination of the special ones. Thus

$$d\left[\frac{1}{(a^2-2x^2)^2}\right] = d(a^2-2x^2)^{-2} = 8(a^2-2x^2)^{-3}x\,dx.$$

Again,

$$d\left[\frac{1}{\sqrt{(a+x)^3}}\right] = d(a+x)^{-\frac{3}{2}} = -\tfrac{3}{2}(a+x)^{-\frac{5}{2}}dx.$$

The derivative of a function may be written at once instead of first writing the differential, since the former differs from the latter only in the omission of the factor dx, which must necessarily occur in every term. Thus, given

$$y = \frac{x}{\sqrt{(1+x^2)}} = x(1+x^2)^{-\frac{1}{2}},$$

we derive

$$\frac{dy}{dx} = (1+x^2)^{-\frac{1}{2}} - \tfrac{1}{2}x(1+x^2)^{-\frac{3}{2}}\,.\,2x = \frac{1}{(1+x^2)^{\frac{3}{2}}}.$$

Examples V

1. Differentiate $\dfrac{a+bx+cx^2}{x}$.

Put the expression in the form $\dfrac{a}{x}+b+cx$. $\qquad \left(c-\dfrac{a}{x^2}\right)dx.$

Find the derivatives of the following functions:

2. $y = \dfrac{a^2-b^2}{a^2-x^2}$. $\qquad \dfrac{dy}{dx} = (a^2-b^2)\dfrac{2x}{(a^2-x^2)^2}$.

3. $y = \sqrt{(x^3-a^3)}$. $\qquad \dfrac{dy}{dx} = \dfrac{3x^2}{2\sqrt{(x^3-a^3)}}$.

4. $y = \dfrac{2x^4}{a^2 - x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{4x^3(2a^2 - x^2)}{(a^2 - x^2)^2}$.

5. $y = (1 + 2x^2)(1 + 4x^3)$. $\qquad \dfrac{dy}{dx} = 4x(1 + 3x + 10x^3)$.

6. $y = (a^3 + x^3)(b^2 + 3x^2)$ $\qquad \dfrac{dy}{dx} = 3(5x^3 + bx^2 + 2a^3)x$.

7. $y = (1 + x)^4(1 + x^2)^2$.

$$\frac{dy}{dx} = 4(1 + x)^3(1 + x^2)(1 + x + 2x^2).$$

8. $y = (1 + x^m)^n + (1 + x^n)^m$.

$$\frac{dy}{dx} = mn[(1 + x^m)^{n-1}x^{m-1} + (1 + x^n)^{m-1}x^{n-1}].$$

9. $y = \dfrac{x^2 - 2a^2}{x - a}$. $\qquad \dfrac{dy}{dx} = 1 + \dfrac{a^2}{(x - a)^2}$.

10. $y = \dfrac{a - x}{\sqrt{x}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{a + x}{2x^{\frac{3}{2}}}$.

11. $y = \dfrac{\sqrt{(x^2 - a^2)}}{x}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^2}{x^2\sqrt{(x^2 - a^2)}}$.

12. $y = \dfrac{ab}{cx\sqrt{(x^2 - a^2)}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{ab}{c} \cdot \dfrac{2x^2 - a^2}{x^2(x^2 - a^2)^{\frac{3}{2}}}$.

13. $y = \dfrac{1}{\sqrt{(1 + x)}} + \dfrac{1}{\sqrt{(1 - x)}}$.

$$\frac{dx}{dx} = \tfrac{1}{2}[(1 - x)^{-\frac{3}{2}} - (1 + x)^{-\frac{3}{2}}].$$

14. $y = (1 + x)\sqrt{(1 - x)}$. $\qquad \dfrac{dy}{dx} = \dfrac{1 - 3x}{2\sqrt{(1 - x)}}$.

15. $y = (a + x)^3(b - x)^4x^2$.

$$\frac{dy}{dx} = x(a + x)^2(b - x)^3[2ab + (5b - 6a)x - 9x^2].$$

16. $y = \dfrac{x^n + 1}{x^n - 1}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2nx^{n-1}}{(x^n - 1)^2}$.

17. $y = (3b + 2ax)^{\frac{3}{2}}(b - ax)$. $\qquad \dfrac{dy}{dx} = -5a^2x\sqrt{(3b + 2ax)}$.

18. $y = \sqrt{\dfrac{1+x}{1-x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{(1-x)\sqrt{(1-x^2)}}.$

19. $y = \dfrac{x}{\sqrt{(a^2+x^2)} - x}.$

Rationalize the denominator. $\qquad \dfrac{dy}{dx} = \dfrac{1}{a^2}\left[\dfrac{a^2+2x^2}{\sqrt{(a^2+x^2)}} + 2x\right].$

20. $y = \dfrac{x}{\sqrt{(a^2-x^2)}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{a^2}{(a^2-x^2)^{\frac{3}{2}}}.$

21. $y = \dfrac{bx}{\sqrt{(2ax-x^2)}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{abx}{(2ax-x^2)^{\frac{3}{2}}}.$

22. $y = \dfrac{a^2-b^2}{(2ax-x^2)^{\frac{3}{2}}}.$

See Art. 55. $\qquad \dfrac{dy}{dx} = 3(a^2-b^2)\dfrac{x-a}{(2ax-x^2)^{\frac{5}{2}}}.$

23. $y = \dfrac{x\sqrt{(a+x)}}{\sqrt{a} - \sqrt{(a-x)}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a}}{2\sqrt{(a+x)}} - \dfrac{x}{\sqrt{(a^2-x^2)}}.$

24. $y = \dfrac{x^3}{\sqrt{(1-x^3)}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{3x^2}{2}\cdot\dfrac{2-x^3}{(1-x^3)^{\frac{3}{2}}}.$

25. $y = \dfrac{x^3}{(1-x^2)^{\frac{3}{2}}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{3x^2}{(1-x^2)^{\frac{5}{2}}}.$

26. $y = \dfrac{x^n}{(1+x)^n}.$ $\qquad \dfrac{dy}{dx} = \dfrac{nx^{n-1}}{(1+x)^{n+1}}.$

27. $y = \dfrac{1}{(a+x)^m(b+x)^n}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{na+mb+(m+n)x}{(a+x)^{m+1}(b+x)^{n+1}}.$

28. $y = \dfrac{2x^2-1}{x\sqrt{(1+x^2)}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1+4x^2}{x^2(1+x^2)^{\frac{3}{2}}}.$

29. $y = \dfrac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{x}.$

$$\frac{dy}{dx} = -\frac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{x^2\sqrt{(1-x^4)}}.$$

30. $y = \sqrt{\dfrac{1-x^2}{(1+x^2)^3}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2x^3-4x}{(1-x^2)^{\frac{1}{2}}(1+x^2)^{\frac{5}{2}}}.$

31. $y = x(a^2 + x^2)\sqrt{(a^2 - x^2)}.$ $\qquad \dfrac{dy}{dx} = \dfrac{a^4 + a^2x^2 - 4x^4}{\sqrt{(a^2 - x^2)}}.$

32. $y = \dfrac{x^{2n}}{(1 + x^2)^n}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2nx^{2n-1}}{(1 + x^2)^{n+1}}.$

33. $y = \dfrac{1 - x}{\sqrt{(1 + x^2)}}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{1 + x}{(1 + x^2)^{\frac{3}{2}}}.$

34. $y = \dfrac{x^3}{x + \sqrt{(1 + x^2)}}.$

See Example 19. $\qquad \dfrac{dy}{dx} = \dfrac{4x^4 + 3x^2}{\sqrt{(x^2 + 1)}} - 4x^3.$

35. $y = \dfrac{\sqrt{(1 + x^2)} + \sqrt{(1 - x^2)}}{\sqrt{(1 + x^2)} - \sqrt{(1 - x^2)}}.$

$$\frac{dy}{dx} = -\frac{2}{x^3}\left[1 + \frac{1}{\sqrt{(1 - x^4)}}\right].$$

36. $y = \dfrac{x}{\sqrt{(x^2 + a^2)} - a}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{a}{x^2}\left[1 + \dfrac{a}{\sqrt{(x^2 + a^2)}}\right].$

37. Two locomotives are moving along two straight lines of railway which intersect at an angle of 60°; one is approaching the intersection at the rate of 25 miles an hour, and the other is receding from it at the rate of 30 miles an hour; find the rate per hour at which they are separating from each other when each is 10 miles from the intersection. $2\frac{1}{2}$ miles.

38. A street-crossing is 10 ft. from a street-lamp situated directly above the curbstone, which is 60 ft. from the vertical walls of the opposite buildings. If a man is walking across to the opposite side of the street at the rate of 4 miles an hour, at what rate per hour does his shadow move upon the walls: (α) when he is 5 ft. from the curbstone? (β) when he is 20 ft. from the curbstone?

(α) 96 miles; (β) 6 miles.

39. Assuming the volume of a tree to be proportional to the cube of its diameter, and that the latter increases uniformly, find the ratio of the rate of its volume when the diameter is 6 inches to the rate when the diameter is 3 ft. $\frac{1}{36}$.

40. If an ingot of silver in the form of a parallelopiped expands $\frac{1}{1000}$ part of each of its linear dimensions for each degree of temperature, at what rate per degree of temperature is its volume increasing when the sides are respectively 2, 3 and 6 inches?

If x denote a side, dx may be assumed to denote the rate per degree of temperature. $\frac{27}{250}$ of a cubic inch

VI.

Differentiation of the Logarithmic Function.

56. The logarithm of x to the base b is the value of y in the equation $x = b^y$ and is denoted by $\log_b x$. In finding its derivative regarded as a function of x, we shall employ the method illustrated in Art. 44 in the case of the square.

Assuming another value z of the independent variable connected with x by the relation

$$z = mx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where m is a constant, the fundamental property of the function is expressed by

$$\log z = \log m + \log x, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

the symbol for the base being omitted for the present. Differentiation of equations (1) and (2) gives

$$dz = m\,dx, \qquad d(\log z) = d(\log x);$$

whence, by division, we have

$$\frac{d(\log z)}{dz} = \frac{1}{m}\frac{d(\log x)}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

for the relation between the values of the derivative corresponding to two values of the independent variable connected together by equation (1).

Now, eliminating m by means of equation (1), we obtain

$$z\frac{d(\log z)}{dz} = x\frac{d(\log x)}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

in which the variables z and x are entirely independent, because we arrive at the same result, no matter what the value of m. In other words, we have shown that, if $f(x) = \log x$,

$$zf'(z) = xf'(x).^*$$

It follows, as in Art. 16, that each member of this equation has a value independent of x. Hence we write

$$x\frac{d(\log_b x)}{dx} = B, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

in which we have denoted the "constant" by B, because, while it is independent of x, it is obviously not independent of the base b of the system of logarithms.

57. We have next to derive a functional relation between B and b. For this purpose, let a be another value of the base. From equation (5)

$$d(\log_b x) = \frac{Bdx}{x}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and therefore also

$$d(\log_a x) = \frac{Adx}{x}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

---

* In the graph of the function $y = \log x$, this equation signifies that the value of tan ϕ, or the gradient of the curve, is inversely proportional to the abscissa. See Fig. 9, p. 53.

in which A is the same function of a that B is of b.

Since by the definition of the logarithm

$$x = b^{\log_b x},$$

we have, by taking logarithms to the base a,

$$\log_a x = \log_a b \,.\, \log_b x; \quad . \quad . \quad . \quad . \quad . \quad (8)$$

whence, differentiating by equations (6) and (7),

$$\frac{A dx}{x} = \log_a b \frac{B dx}{x},$$

or

$$A = B \log_a b.$$

Hence, by the properties of logarithms, $A = \log_a b^B$, and

$$b^B = a^A.$$

The independent quantities b and a are here separated, so that, as in Art. 16, the common value of the members is independent of a or b. In other words, it is an absolute constant, and denoting it by e we have

$$b^B = e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now, taking e as the base of a system of logarithms, we have

$$B \log_e b = 1; \qquad \text{whence} \qquad B = \frac{1}{\log_e b}.$$

Finally, substituting in equation (6),

$$d(\log_b x) = \frac{dx}{x \log_e b}. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Napierian Logarithms.

58. The constant e is an incommensurable quantity second only in importance to the constant π. It is known as the *Napierian base*, and the corresponding system of logarithms as the *natural* or *Napierian* system. The method of computing its value to any required degree of accuracy will be found in a subsequent chapter.

Putting e in the place of b in the general formula (b), we have the special case

$$d(\log_e x) = \frac{dx}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b')$$

Thus the natural logarithm is that which has the simplest derivative.* On this account the logarithms employed in analytical investigations are almost exclusively Napierian. Whenever it is necessary, for the purpose of obtaining numerical results, these logarithms may be expressed in terms of the common tabular logarithms by means of the formula

$$\log_{10} x = \log_{10} e \log_e x,$$

which is derived from equation (8), Art. 57, by writing 10 for a and e for b. The value of the constant $\log_{10} e$ will be computed in a subsequent chapter.

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* The ground upon which Napier, the inventor of logarithms, chose the natural base is equivalent to the assumption that x and $\log x$ shall, in starting from their initial values 1 and 0, begin to vary at the same rate.

Hereafter, whenever the symbol log is employed without the subscript, $\log_e$ is to be understood.

59. The graph of the function $\log x$, or curve whose rectangular equation is

$$y = \log_e x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is called the *logarithmic curve*.

The shape of this curve is indicated in Fig. 9. It passes through the point A whose coordinates are $(1, 0)$, since

$$\log 1 = 0.$$

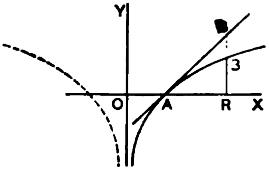

Fig. 9.

Since we have, from formula (b'),

$$\tan\phi = \frac{dy}{dx} = \frac{1}{x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the value of $\tan\phi$ at the point A is unity, and therefore AC, the tangent line at this point, cuts the axis of x at an angle of 45°, as in the diagram. It follows from equation (2) that

when $\quad x > 1, \quad \tan\phi < 1,$

and when $\quad x < 1, \quad \tan\phi > 1;$

the curve, therefore, lies below the tangent AC, as shown in Fig. 9.

The point $(e, 1)$ is a point of the curve; let B, Fig. 9, be this point, then OR will represent the Napierian base, and $BR = 1 = OA$. Produce BR to meet the tangent in C; then, because the tangent lies above the curve, $RC > 1$. But since

$RAC = 45°$, $AR = RC$; hence $AR > 1$ and $OR > 2$; that is, the Napierian base e is somewhat greater than 2.

Logarithmic Differentiation.

60. The expression $\frac{dx}{x}$ is often called the *logarithmic differential* of x. When the value of x is positive, it is by the preceding articles the differential of the Napierian logarithm of x. But, when x is negative, so that the logarithm is imaginary, the logarithmic differential is still real, and is in fact then the differential of the logarithm of the numerical value of x taken positively: for

$$d[\log(-x)] = \frac{d(-x)}{-x} = \frac{dx}{x}.^*$$

Hence we may define the logarithmic differential as *the differential of the logarithm of the numerical value of x* regardless of algebraic sign.

The complete graph of the function corresponding to the logarithmic differential would consist of the curve of Art. 59, together with the dotted branch represented in Fig. 9.

61. By the process of logarithmic differentiation we may derive independent demonstrations of the formulæ already found. Thus, by differentiating the equation

$$\log(xy) = \log x + \log y,$$

---

[*] The logarithm of a negative quantity is imaginary; but, by the properties of logarithms, we must have $\log(-x) = \log x + \log(-1)$. The term $\log(-1)$ constitutes the imaginary part; but, since it is a constant, the differential of $\log(-x)$ is the same as that of $\log x$.

we have

$$\frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y};$$

whence

$$d(xy) = ydx + xdy.$$

In like manner, from

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

we derive

$$\frac{y}{x}d\left(\frac{x}{y}\right) = \frac{dx}{x} - \frac{dy}{y};$$

whence

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}.$$

Again, from

$$\log x^n = n \log x$$

we have

$$\frac{d(x^n)}{x^n} = n\frac{dx}{x};$$

whence

$$d(x^n) = nx^{n-1}dx.$$

62. The method of logarithmic differentiation may frequently be used with advantage in finding the derivatives of complicated algebraic expressions. For example, let us take

$$u = \frac{\sqrt{(2x)}(1 - x^2)^{\frac{3}{4}}}{(x - 2)^{\frac{2}{3}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence we derive

$$\log u = \tfrac{1}{2}\log(2x) + \tfrac{3}{4}\log(1 - x^2) - \tfrac{2}{3}\log(x - 2). \quad (2)$$

Differentiating,

$$\frac{du}{udx} = \frac{1}{2x} - \frac{3x}{2(1 - x^2)} - \frac{2}{3(x - 2)}; \quad . \ . \ . \quad (3)$$

adding and reducing,

$$\frac{du}{udx} = \frac{-8x^3 + 24x^2 - x - 6}{6(1 - x^2)(x - 2)x};$$

therefore

$$\frac{du}{dx} = \frac{-8x^3 + 24x^2 - x - 6}{3(2x)^{\frac{1}{2}}(1 - x^2)^{\frac{1}{4}}(x - 2)^{\frac{5}{3}}}.$$

Differentiation of Exponential Functions.

63. An exponential function is an expression in which the exponent is variable. In the simple exponential function

$$y = a^x \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

it is necessary to assume a to be positive; otherwise y is not a continuous variable. This is the same thing as saying that in the inverse of this function, $x = \log_a y$, we cannot have a negative base for a system of logarithms.

The differential may be obtained from the inverse function, as in Art. 46, or as follows, which amounts to the same thing. Taking Napierian logarithms of both members of equation (1), we have

$$\log y = x \log a;$$

differentiating by formula (b'),

$$\frac{dy}{y} = \log a \,.\, dx;$$

hence

$$dy = \log a \,.\, y\,dx,$$

or

$$d(a^x) = \log a \,.\, a^x dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

64. Putting e for a in this formula we have the special case

$$d(e^x) = e^x dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c')$$

The exponential of this form is the most common in analysis. It is often denoted by the functional symbol exp, especially when the exponent is a complicated expression, for example, $\exp[x\sqrt{(x^2 - 1)}]$.

Formula (c') shows that e^x is the function which is its own derivative. Its graph, Fig. 10, is called *the exponential curve*, and is the same as the logarithmic curve in another position. If from any point P of a curve the tangent PT and ordinate PR be drawn, the subtangent $TR = y \cot \phi$. Now in this curve, $\tan \phi = e^x = y$, we therefore have $y \cot \phi = 1$. Hence the exponential curve is the curve in which the subtangent is constant. In the diagram, this constant value is equal to OB, which represent unity.

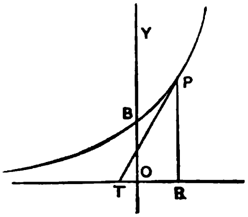

FIG. 10.

65. When both the exponent and the quantity affected by it are variable, the method of logarithmic differentiation may be employed. Thus, if the given function be

$$z = (nx)^{x^n},$$

we shall have

$$\log z = x^2 \log (nx);$$

differentiating,

$$\frac{dz}{z} = x^2 \frac{dx}{x} + 2x \log (nx) dx.$$

hence

$$d[(nx)^{x^2}] = (nx)^{x^2} x[1 + 2 \log (nx)] dx.$$

Examples VI.

1. Given the function $y = \log_b x$; show that $\left.\frac{dy}{dx}\right]_e = \frac{\log_b e}{e}$, and hence prove that the tangent to the corresponding curve, at the point whose abscissa is e, passes through the origin.

2. $y = x^n \log x$. $\qquad \frac{dy}{dx} = x^{n-1}(1 + n \log x)$.

3. $y = \log (\log x)$. $\qquad \frac{dy}{dx} = \frac{1}{x \log x}$.

4. $y = \log [\log(a + bx^n)]$. $\qquad \frac{dy}{dx} = \frac{nbx^{n-1}}{(a + bx^n) \log (a + bx^n)}$.

5. $y = \sqrt{x} - \log (\sqrt{x} + 1)$. $\qquad \frac{dy}{dx} = \frac{1}{2(\sqrt{x} + 1)}$.

6. $y = \log \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}$. $\qquad \frac{dy}{dx} = \frac{\sqrt{a}}{(a - x)\sqrt{x}}$.

Put in the form $\log (\sqrt{a} + \sqrt{x}) - \log (\sqrt{a} - \sqrt{x})$.

7. $y = \log [(\sqrt{x - a}) + \sqrt{(x - b)}]$. $\qquad \frac{dy}{dx} = \frac{1}{2\sqrt{[(x-a)(x-b)]}}$.

8. $y = \log [x + \sqrt{(x^2 \pm a^2)}]$. $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{(x^2 \pm a^2)}}$.

9. $y = \log \frac{x}{\sqrt{(1 + x^2)}}$. $\qquad \frac{dy}{dx} = \frac{1}{x(1 + x^2)}$.

10. $y = \log \frac{\sqrt{(1 + x)} + \sqrt{(1 - x)}}{\sqrt{(1 + x)} - \sqrt{(1 - x)}}$. $\qquad \frac{dy}{dx} = -\frac{1}{x\sqrt{(1 - x^2)}}$.

11. $y = \log [x + \sqrt{(a^2 - x^2)}]$. $\qquad \frac{dy}{dx} = \frac{\sqrt{(a^2 - x^2)} - x}{\sqrt{(a^2 - x^2)}[x + \sqrt{(a^2 - x^2)}]}$.

12. $y = \log \dfrac{x}{\sqrt{(x^2 + a^2)} - x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{\sqrt{(x^2 + a^2)}}$.

13. $y = \log[\sqrt{(1+x^2)} + \sqrt{(1-x^2)}]$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{x\sqrt{(1-x^4)}}$.

14. $y = \log(x-a) - \dfrac{a(2x-a)}{(x-a)^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^2 + a^2}{(x-a)^3}$.

15. $y = a^{x^2}$. $\qquad \dfrac{dy}{dx} = 2 \log a \, a^{x^2} x$.

16. $y = e^{\frac{1}{1+x}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2} \cdot e^{\frac{1}{1+x}}$.

17. $y = e^x(1 - x^3)$. $\qquad \dfrac{dy}{dx} = e^x(1 - 3x^2 - x^3)$.

18. $y = (x-3)e^{2x} + 4xe^x$. $\qquad \dfrac{dy}{dx} = (2x-5)e^{2x} + 4(x+1)e^x$.

19. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{4}{(e^x + e^{-x})^2}$.

20. $y = b^{a^x}$. $\qquad \dfrac{dy}{dx} = \log a \, . \log b \, . \, b^{a^x} . \, a^x$.

21. $y = a^{x^n}$. $\qquad \dfrac{dy}{dx} = na^{x^n} . \, x^{n-1} . \log a$.

22. $y = \dfrac{x}{e^x - 1}$. $\qquad \dfrac{dy}{dx} = \dfrac{e^x(1-x) - 1}{(e^x - 1)^2}$.

23. $y = \log(e^x + e^{-x})$. $\qquad \dfrac{dy}{dx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

24. $y = a^{\log x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x} \log a \, . \, a^{\log x}$.

25. $y = \log \dfrac{e^x}{1 + e^x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{1 + e^x}$.

26. $y = x^x$. $\qquad \dfrac{dy}{dx} = x^x(1 + \log x)$.

27. $y = x^{\log x}$. $\qquad \dfrac{dy}{dx} = \dfrac{2 \log x}{x} x^{\log x}$.

28. $y = e^{x^x}$. $\qquad \dfrac{dy}{dx} = e^{x^x} . \, x^x(1 + \log x)$.

29. $y = x^{\frac{1}{x}}$. $\qquad \dfrac{dy}{dx} = x^{\frac{1}{x}} \dfrac{1 - \log x}{x^2}$.

30. $y = e^{e^x}$. $\qquad \dfrac{dy}{dx} = e^{e^x} . e^x$.

31. $y = xe^{x}$. $\qquad \dfrac{dy}{dx} = x^{e^x} . e^x \dfrac{1 + x \log x}{x}$.

32. $y = a^x (x \log a - 1)$. $\qquad \dfrac{dy}{dx} = (\log a)^2 x a^x$.

33. $y = 2e^{\sqrt{x}}(x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6)$. $\qquad \dfrac{dy}{dx} = xe^{\sqrt{x}}$.

34. $y = \dfrac{(x-1)^{\frac{5}{2}}}{(x-2)^{\frac{3}{4}}(x-3)^{\frac{7}{3}}}$.

See Art. 62. $\qquad \dfrac{dy}{dx} = -\dfrac{(x-1)^{\frac{3}{2}}(7x^2 + 30x - 97)}{12(x-2)^{\frac{7}{4}}(x-3)^{\frac{10}{3}}}$.

35. $y = \dfrac{\sqrt{[ax(x-3a)]}}{\sqrt{(x-4a)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a}(x^2 - 8ax + 12a^2)}{2[x(x-3a)]^{\frac{1}{2}}(x-4a)^{\frac{3}{2}}}$.

36. $y = \dfrac{(x+1)^{\frac{1}{2}}(x+3)^{\frac{9}{2}}}{(x+2)^4}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^2(x+3)^{\frac{7}{2}}}{(x+2)^5(x+1)^{\frac{1}{2}}}$.

37. $y = \dfrac{(x-2)^9}{(x-1)^{\frac{5}{2}}(x-3)^{\frac{11}{2}}}$. $\qquad \dfrac{dy}{dx} = \dfrac{(x-2)^8(x^2 - 7x + 1)}{(x-1)^{\frac{7}{2}}(x-3)^{\frac{13}{2}}}$.

38. $y = \dfrac{(x^2 - 2x + 2)^{\frac{1}{4}}(x^2+1)^{\frac{3}{2}}}{(x+1)^{\frac{7}{2}}}$.

$$\frac{dy}{dx} = \frac{(8x^3 - 21x^2 + 26x - 15)(x^2+1)^{\frac{1}{2}}}{2(x^2 - 2x + 2)^{\frac{3}{4}}(x+1)^{\frac{9}{2}}}.$$

39. $y = \dfrac{\exp[x\sqrt{(x^2-1)}]}{x + \sqrt{(x^2-1)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{2\sqrt{(x^2-1)}\exp[x\sqrt{(x^2-1)}]}{x + \sqrt{(x^2-1)}}$.

40. Given $y = \left(\dfrac{x}{1 + \sqrt{(1-x^2)}}\right)^n$, prove $\dfrac{dy}{dx} = \dfrac{ny}{x\sqrt{(1-x^2)}}$.

41. Given $u = \dfrac{x}{\sqrt{(1-x^2)}}\left(\dfrac{x}{1 + \sqrt{(1-x^2)}}\right)^n$, prove that

$$\frac{du}{dx} = \left(\frac{x}{1 + \sqrt{(1-x^2)}}\right)^n \frac{1 + n\sqrt{(1-x^2)}}{(1-x^2)^{\frac{3}{2}}}.$$

Put $\left(\dfrac{x}{1 + \sqrt{(1-x^2)}}\right)^n = y$, *and use the result obtained in Ex.* 40.

VII.

Trigonometric or Circular Functions.

66. As stated in Art. 7, when the trigonometric functions, sin θ, cos θ etc., are regarded as undergoing continuous variation, the independent variable θ is taken to be the arcual measure of the angle, that is, the ratio which the arc subtending the angle at the centre of a circle bears to the radius.

Let O be the centre of a circle of radius a, referred to rectangular diameters as coordinate axes, and let AOP, Fig. 11, be the angle θ, the radius OA being fixed in the axis of x. Denote the arc AP by s, then $\theta = s/a$, and as θ increases P moves along the circumference, completing the circuit (and the radius OP completing a revolution), when $\theta = 2\pi$. Since s may increase indefinitely with repeated revolutions of P, the trigonometric functions defined by the ratios of the sides of the right triangle OPR are continuous functions for all values of θ. They are called *periodic functions*, because their values repeat themselves while θ passes through successive ranges of values, each of extent 2π; hence also 2π is called *the period* of the functions. Compare the graphs in Figs. 12 and 13, pp. 63 and 64.

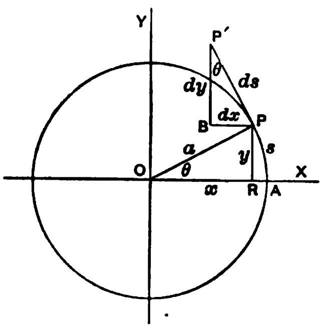

FIG. 11.

Since both the functions and the independent variable are defined by means of lines connected with a circle, they are often called *circular functions*. We shall derive their rates of variation from their geometrical definitions.

Differentiation of the Sine and the Cosine.

67. We have then, by definition, in Fig. 11

$$\theta = \frac{s}{a}, \qquad \sin\theta = \frac{y}{a}, \qquad \cos\theta = \frac{x}{a}. \quad . \quad . \quad . \quad (1)$$

Let PP', measured along the tangent line in the direction in which P moves when θ is increasing, represent ds; then drawing PB and BP' parallel to the axes, we complete, as in Fig. 7, *the differential triangle* for the motion of P. From their directions in Fig. 11 it appears that BP' represents dy, which is positive because y is increasing, and BP represents $-dx$ because x is decreasing, so that dx is negative.

Differentiating equations (1), we have

$$d\theta = \frac{ds}{a}, \qquad d(\sin\theta) = \frac{dy}{a}, \qquad d(\cos\theta) = \frac{dx}{a};$$

whence the derivatives of $\sin\theta$ and $\cos\theta$ are

$$\frac{d(\sin\theta)}{d\theta} = \frac{dy}{ds}, \qquad \frac{d(\cos\theta)}{d\theta} = \frac{dx}{ds}. \quad . \quad . \quad . \quad (2)$$

To express these in terms of θ, we note that the differential triangle and OPR are similar because their sides are mutually perpendicular, the tangent to a circle being perpendicular to the radius drawn to the point of contact.*

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* This can be shown independently of geometry by differentiating the equation of the circle, which is

$$x^2 + y^2 = a^2;$$

for this gives

$$x\,dx + y\,dy = 0,$$

whence

$$\frac{dy}{-dx} = \frac{x}{y}.$$

Therefore the angle *BPP* is equal to θ, and

$$\frac{dy}{ds} = \cos\theta, \qquad \frac{dx}{ds} = -\sin\theta.$$

Substituting in equation (2), we have

$$d(\sin\theta) = \cos\theta\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

and

$$d(\cos\theta) = -\sin\theta\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

68. The arcual measure is the length of the arc in the circle whose radius is unity. In the graph of the function $\sin x$, this is measured along the axis of x. The curve $y = \sin x$ is the full line in Fig. 12, which is a continuous curve of unlimited extent, consisting of repeated similar branches. Since

$$\frac{d(\sin x)}{dx}\Big]_0 = \cos 0 = 1,$$

this curve makes an angle of 45° with the axis of x at the origin. The graph of $\cos x$ is the dotted line in the diagram, which is the same curve moved a distance equal $\frac{1}{2}\pi$ to the left, since $\cos x = \sin(x + \frac{1}{2}\pi)$.

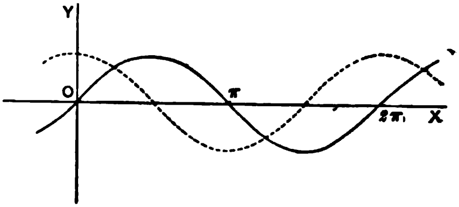

FIG. 12.

Differentiation of the Tangent and the Cotangent.

69. The differential of $\tan\theta$ is found by applying formula (C), p. 42, to the equation

$$\tan\theta = \frac{\sin\theta}{\cos\theta};$$

thus, using formulæ (*d*) and (*e*),

$$d(\tan\theta) = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\,d\theta = \frac{d\theta}{\cos^2\theta},$$

or

$$d(\tan\theta) = \frac{d\theta}{\cos^2\theta} = \sec^2\theta\,d\theta. \quad . \quad . \quad . \quad . \quad (f)$$

The differential of cot θ can be obtained in a similar manner, or by applying this formula to the equation

$$\cot\theta = \tan(\tfrac{1}{2}\pi - \theta);$$

which gives

$$d(\cot\theta) = -\frac{d\theta}{\sin^2\theta} = -\operatorname{cosec}^2\theta\,d\theta. \quad . \quad . \quad (g)$$

70. The function $\tan x$ is discontinuous for any range of values including an odd multiple of $\frac{1}{2}\pi$, because the value of the function is infinite when $x = \pm\frac{1}{2}\pi$, $\pm\frac{3}{2}\pi$, etc. Accordingly, the graph or curve $y = \tan x$, given in Fig. 13, consists of an unlimited number of detached branches, each corresponding to a range of values of x of extent π. The period of this function is therefore π. Each branch cuts the axis of x at an angle of 45°, and at these points the curve has, by formula (f), its smallest gradient.

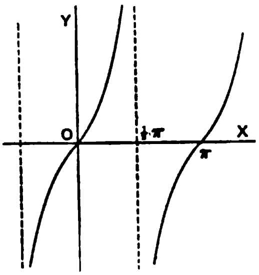

FIG. 13.

Differentiation of the Secant and the Cosecant.

71. The differential of sec θ is found by applying the rule for the reciprocal to the equation

$$\sec\theta = \frac{1}{\cos\theta};$$

thus

$$d(\sec\theta) = \frac{\sin\theta\, d\theta}{\cos^2\theta} = \sec\theta\tan\theta\, d\theta. \quad . \quad . \quad . \quad (h)$$

The differential of cosec θ is found by applying this formula to the equation

$$\operatorname{cosec}\theta = \sec(\tfrac{1}{2}\pi - \theta),$$

which gives

$$d(\operatorname{cosec}\theta) = -\cot^2\theta\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad (i)$$

The graph of the function sec x is given in Fig. 14. It consists of discontinuous branches alternately above and below the axis of x. The period of the function, 2π, corresponds to two branches.

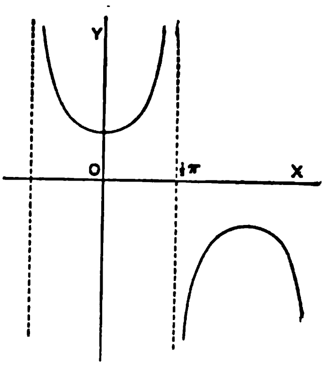

FIG. 14.

72. To these formulæ (d) to (i) for the six trigonometric functions may be added that for the *versed-sine*, which in Fig. 11 is the ratio of AR to the radius OP. This function is therefore defined by the equation

$$\operatorname{vers}\theta = 1 - \cos\theta,$$

whence, by formula (e),

$$d(\operatorname{vers}\theta) = \sin\theta\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (j)$$

The Logarithmic Trigonometric Functions.

73. Combining the formulæ found above with that for the logarithm, we readily derive the following for the logarithms of the circular functions:

$$d(\log \sin \theta) = -\,d(\log \operatorname{cosec} \theta) = \cot \theta \, d\theta;$$

$$d(\log \cos \theta) = -\,d(\log \sec \theta) = -\tan \theta \, d\theta;$$

$$d(\log \tan \theta) = -\,d(\log \cot \theta) = (\tan \theta + \cot \theta)\, d\theta.$$

Examples VII.

1. The value of $d(\sin \theta)$ being given, derive that of $d(\cos \theta)$ from the identity $\cos^2\theta = 1 - \sin^2\theta$.

2. From the identity $\sec^2\theta = 1 + \tan^2\theta$, derive the differential of $\sec \theta$.

3. From the identity $\sin 2\theta = 2 \sin \theta \cos \theta$, derive another by taking derivatives. $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Find the derivatives of the following functions:

4. $y = \theta + \sin \theta \cos \theta$. $\qquad \dfrac{dy}{d\theta} = 2 \cos^2\theta$.

5. $y = \sin \theta - \frac{1}{3} \sin^3\theta$. $\qquad \dfrac{dy}{d\theta} = \cos^3\theta$.

6. $y = \dfrac{\sin \theta}{\sqrt{(\cos \theta)}}$. $\qquad \dfrac{dy}{d\theta} = \dfrac{1 + \cos^2\theta}{2\,(\cos \theta)^{\frac{3}{2}}}$.

7. $y = \dfrac{\sin x}{x}$. $\qquad \dfrac{dy}{dx} = \dfrac{x \cos x - \sin x}{x^2}$.

8. $y = \sin^2 2x$. $\qquad \dfrac{dy}{dx} = 2 \sin 4x$.

9. $y = \sin^3 x + \cos^3 x$. $\qquad \dfrac{dy}{dx} = 3 \sin x \cos x (\sin x - \cos x)$.

10. $y = \frac{1}{3} \tan^3\theta - \tan \theta + \theta$. $\qquad \dfrac{dy}{d\theta} = \tan^4\theta$.

11. $y = \frac{1}{3} \tan^3\theta + \tan \theta$. $\qquad \dfrac{dy}{d\theta} = \sec^4\theta$.

12. $y = \sin e^x$. $\qquad \dfrac{dy}{dx} = e^x \cos e^x$.

13. $y = x \sin x^2$. $\qquad \dfrac{dy}{dx} = \sin x^2 + 2x^2 \cos x^2$.

14. $y = a^{\sin x}$. $\qquad \dfrac{dy}{dx} = \log a \,.\, a^{\sin x} \cos x$.

15. $y = \tan^2\theta + \log(\cos^2\theta)$. $\qquad \dfrac{dy}{d\theta} = 2 \tan^3\theta$.

16. $y = \log(\tan\theta + \sec\theta)$. $\qquad \dfrac{dy}{d\theta} = \sec\theta$.

17. $y = \log \tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$. $\qquad \dfrac{dy}{d\theta} = \dfrac{1}{\cos\theta}$.

18. $y = x + \log \cos(\frac{1}{4}\pi - x)$. $\qquad \dfrac{dy}{dx} = \dfrac{2}{1 + \tan x}$.

19. $y = \log \sqrt{(\sin x)} + \log \sqrt{(\cos x)}$. $\qquad \dfrac{dy}{dx} = \cot 2x$.

20. $y = \sin n\theta (\sin\theta)^n$. $\qquad \dfrac{dy}{d\theta} = n(\sin\theta)^{n-1} \sin(n+1)\theta$.

21. $y = \dfrac{\sin x}{1 + \tan x}$. $\qquad \dfrac{dy}{dx} = \dfrac{\cos^3 x - \sin^3 x}{(\sin x + \cos x)^2}$.

22. $y = e^{ax} \cos bx$. $\qquad \dfrac{dy}{dx} = e^{ax}(a \cos bx - b \sin bx)$.

23. $y = \log \sqrt{\dfrac{a \cos x - b \sin x}{a \cos x + b \sin x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{-ab}{a^2 \cos^2 x - b^2 \sin^2 x}$.

24. $y = e^x(\cos x - \sin x)$. $\qquad \dfrac{dy}{dx} = -2e^x \sin x$.

25. $y = \tan e^{\frac{1}{x}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{e^{\frac{1}{x}} \sec^2 e^{\frac{1}{x}}}{x^2}$.

26. $y = e^{ax}(a \sin x - \cos x)$. $\qquad \dfrac{dy}{dx} = (a^2 + 1)e^{ax} \sin x$.

27. $y = \sqrt{(1 + \sin x)}$. $\qquad \dfrac{dy}{dx} = \frac{1}{2}\sqrt{(1 - \sin x)}$.

28. $y = \dfrac{(\sin nx)^m}{(\cos mx)^n}$. $\qquad \dfrac{dy}{dx} = \dfrac{mn (\sin nx)^{m-1} \cos(m-n)x}{(\cos mx)^{n+1}}$.

29. $y = \tan \sqrt{(1 - x)}$. $\qquad \dfrac{dy}{dx} = \dfrac{-\sec^2 \sqrt{(1-x)}}{2\sqrt{(1-x)}}$.

30. $y = x^{\sin x}$. $\qquad \dfrac{dy}{dx} = x^{\sin x}\left(\cos x \,.\, \log x + \dfrac{\sin x}{x}\right)$.

31. $y = \sin(\log nx)$. $\qquad \dfrac{dv}{dx} = \dfrac{\cos(\log nx)}{x}$.

32. $y = \sin(\sin x)$. $\qquad \dfrac{dy}{dx} = \cos x \,.\, \cos(\sin x)$.

33. $y = \dfrac{2}{\sin^2 x \cos x} - \dfrac{3\cos x}{\sin^2 x} + 3\log\tan\dfrac{x}{2}$. $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sin^3 x \cos^2 x}$.

34. The crank of a small steam-engine is 1 foot in length, and revolves uniformly at the rate of two turns per second, the connecting rod being 5 ft. in length; find the velocity per second of the piston when the crank makes an angle of 45° with the line of motion of the piston-rod; also when the angle is 135°, and when it is 90°.

Solution :—

Let a, b and x denote respectively the crank, the connecting-rod and the variable side of the triangle; and let θ denote the angle between a and x.

We easily deduce

$$x = a\cos\theta + \sqrt{(b^2 - a^2\sin^2\theta)};$$

whence

$$\frac{dx}{dt} = -\left(a\sin\theta + \frac{a^2\sin\theta\cos\theta}{\sqrt{(b^2 - a^2\sin^2\theta)}}\right)\frac{d\theta}{dt}.$$

In this case, $\dfrac{d\theta}{dt} = 4\pi$, $a = 1$ and $b = 5$.

When $\theta = 45°$, $\dfrac{dx}{dt} = -\dfrac{16\pi\sqrt{2}}{7}$ ft.

35. An elliptical cam revolves at the rate of two turns per second about a horizontal axis passing through one of the foci, and gives a reciprocating motion to a bar moving in vertical guides in a line with

the centre of rotation: denoting by θ the angle between the vertical and the major axis, find the velocity per second with which the bar is moving when $\theta = 60°$, the eccentricity of the ellipse being $\frac{1}{2}$, and the major semi-axis 9 inches. Also find the velocity when $\theta = 90°$.

The relation between θ and the radius vector is expressed by the equation

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}.$$

$$\text{When } \theta = 60°, \ \frac{dr}{dt} = -12\sqrt{3}\pi \text{ inches.}$$

36. Find an expression in terms of its azimuth for the rate at which the altitude of a star is increasing.

Solution:—

Let h denote the altitude and A the azimuth of the star, p its polar distance, t the hour angle, and L the latitude of the observer; the formulæ of spherical trigonometry give

$$\sin h = \sin L \cos p + \cos L \sin p \cos t \quad . \ . \ . \quad (1)$$

and

$$\sin p \sin t = \sin A \cos h. \ . \ . \ . \ . \ . \ . \quad (2)$$

Differentiating (1), p and L being constant,

$$\cos h \frac{dh}{dt} = -\cos L \sin p \sin t,$$

whence, substituting the value of $\sin p \sin t$, from equation (2),

$$\frac{dh}{dt} = -\cos L \sin A.$$

It follows that $\frac{dh}{dt}$ is greatest when $\sin A$ is numerically greatest; that is, when the star is on the prime vertical. In the case of a star that never reaches the prime vertical, the rate is greatest when A is greatest.

VIII.

The Inverse Circular Functions.

74. The graph of the function $y = \sin^{-1}x$ is given in Fig. 15; it is the same as that of $\sin x$, Fig. 12, with x and y interchanged. It shows that the function is real only for values of x between $+1$ and -1, and that for any value of x between these limits there are an infinite number of values of the function. To distinguish between these, we have marked by the full line a portion of the curve corresponding to one value, and only one, for every admissible value of x. The values of y thus selected are all between $\pm\frac{1}{2}\pi$ and are called *the primary values* of $\sin^{-1}x$. The primary value of $\sin^{-1}x$ may thus be regarded as a one-valued function continuous for the range from $x = -1$ to $x = +1$.

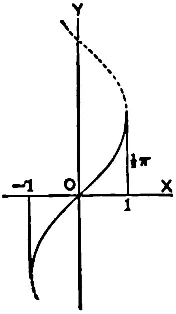

FIG. 15.

When not otherwise stated, the symbol $\sin^{-1}x$ will be considered to mean the primary value. In particular, $\sin^{-1}(-1) = -\frac{1}{2}\pi$, $\sin^{-1}0 = 0$, $\sin^{-1}1 = \frac{1}{2}\pi$. Then, as proved in Trigonometry and illustrated by the figure, the other values of the inverse sine are all included in one or the other of the two expressions

$$2n\pi + \sin^{-1}x \qquad \text{or} \qquad (2n+1)\pi - \sin^{-1}x,$$

where n is a positive or negative integer.

75. The graph of the function $\cos^{-1}x$ is given in Fig. 16. The portion indicated by the full line corresponds to one and only one value of $\cos^{-1}x$ for every admissible value of

x. These values are all between 0 and π, and are called *the primary values* of the function $\cos^{-1}x$. Thus the primary value may be taken as a one-valued continuous function for the range between -1 and $+1$.

When not otherwise stated, $\cos^{-1}x$ will be taken to mean the primary value. The other values are included in the expressions $2n\pi \pm \cos^{-1}x$, where n is an integer. In particular $\cos^{-1}(-1) = \pi$, $\cos^{-1}0 = \frac{1}{2}\pi$, $\cos^{-1}1 = 0$.

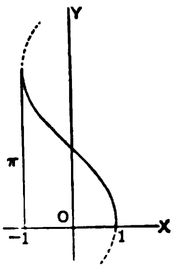

FIG. 16.

For negative as well as for positive values of x, the primary values of $\sin^{-1}x$ and $\cos^{-1}x$ satisfy the relation

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x.$$

76. The graph of the function $\tan^{-1}x$ is the same as Fig. 13 with x and y interchanged. The primary values correspond to that branch which passes through the origin as given in Fig. 17. Since all values of the tangent are possible, $\tan^{-1}x$ is a continuous function for all values of x. The curve approaches the straight lines $y = \pm\frac{1}{2}\pi$ as asymptotes. Thus, using the primary value, we write $\tan^{-1}(-\infty) = -\frac{1}{2}\pi$, $\tan^{-1}0 = 0$, $\tan^{-1}\infty = \frac{1}{2}\pi$.

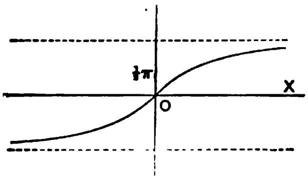

FIG. 17.

The other values of the inverse tangent are all included in the single expression

$$n\pi + \tan^{-1}x,$$

where n is a positive or negative integer.

77. It will be noticed that, for each of the functions $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$, the primary value is so taken that it is in the first quadrant (that is, between 0 and $\frac{1}{2}\pi$) for all admissible positive values of x. Also, the primary value for negative values of x is so taken as to form a function continuous for the whole range of admissible values of x, as shown by the graph in each case.

In the case of the other three inverse circular functions, the same rule applies for positive values of x. It is not, however, possible in the case of $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ to obtain a continuous function for positive and negative values of x, because these functions are not real for values of x between $+1$ and -1. For $\cot^{-1}x$ we can obtain a continuous function by taking the primary value between 0 and π, while x varies from $+\infty$ to $-\infty$.

The use of these last three functions can be avoided by means of the transformations:

$$\sec^{-1}\frac{\alpha}{\beta} = \cos^{-1}\frac{\beta}{\alpha}, \qquad \operatorname{cosec}^{-1}\frac{\alpha}{\beta} = \sin^{-1}\frac{\beta}{\alpha},$$

$$\cot^{-1}\frac{\alpha}{\beta} = \tan^{-1}\frac{\beta}{\alpha}.$$

Differentiation of the Inverse Sine.

78. Proceeding in the usual manner for an inverse function, let

$$y = \sin^{-1}x; \qquad \text{whence} \qquad x = \sin y.$$

Then, by formula (d);

$$dx = \cos y\, dy; \qquad \text{whence} \qquad dy = \frac{dx}{\cos y}.$$

Now $\cos y = \pm\sqrt{(1 - \sin^2 y)} = \pm\sqrt{(1 - x^2)}$, but if y is restricted to the primary value of $\sin^{-1}x$, which lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, $\cos y$ is positive. Hence, substituting, we have for the primary value

$$d(\sin^{-1}x) = \frac{dx}{\sqrt{(1 - x^2)}} \quad . \quad . \quad . \quad . \quad . \qquad (k)$$

Accordingly, the primary value of $\sin^{-1}x$ is a continuous increasing function for the range of values of x between -1 and $+1$, as shown in the graph, Fig. 15, p. 70.

79. Similarly, when

$$y = \cos^{-1}x, \qquad x = \cos y;$$

by formula (*e*),

$$dx = -\sin y\, dy; \qquad \text{whence} \qquad dy = -\frac{dx}{\sin y}.$$

Here $\sin y = \pm\sqrt{(1 - \cos^2 y)} = \pm\sqrt{(1 - x^2)}$; but, when y is restricted to the primary value of $\cos^{-1}x$ which lies between 0 and π, $\sin y$ is positive. Therefore, for the primary value we have

$$d\cos^{-1}x) = -\frac{dx}{\sqrt{(1 - x^2)}}. \quad . \quad . \quad . \quad . \qquad (l)$$

Accordingly, as shown in the graph, Fig. 16, p. 71, the primary value is a decreasing function.

Differentiation of the Inverse Tangent.

80. If $\qquad y = \tan^{-1}x, \qquad x = \tan y,$
and, by formula (*f*),

$$dx = \sec^2 y\, dy; \qquad \text{whence} \qquad dy = \frac{dx}{\sec^2 y}.$$

But $\sec^2 y = 1 + \tan^2 y = 1 + x^2$, therefore

$$d(\tan^{-1}x) = \frac{dx}{1 + x^2}. \quad . \quad . \quad . \quad . \quad . \quad (m)$$

Since $1 + x^2$ is always positive, $\tan^{-1}x$ is an increasing function as shown by the graph, Fig. 17, p. 71.

The expression applies to all the values of $\tan^{-1}x$, which in fact differ only by values of the constant $n\pi$ in the expression given in Art. 76.

In like manner, or from the relation

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x,$$

we derive

$$d(\cot^{-1}x) = -\frac{dx}{1 + x^2}. \quad . \quad . \quad . \quad . \quad (n)$$

Differentiation of the Inverse Secant.

81. If $\qquad y = \sec^{-1}x, \qquad x = \sec y,$
and, by formula (h),

$$dy = \frac{dx}{\sec y \tan y},$$

where $\sec y = x$ and $\tan y = \pm \sqrt{(x^2 - 1)}$. But, taking y in the first quadrant when x is positive, $\tan y$ is positive, therefore for the primary value

$$d(\sec^{-1}x) = \frac{dx}{x\sqrt{(x^2 - 1)}}. \quad . \quad . \quad . \quad . \quad (o)$$

In like manner, or from $\operatorname{cosec}^{-1}x = \frac{1}{2}\pi - \sec^{-1}x$, we derive

$$d(\operatorname{cosec}^{-1}x) = -\frac{dx}{x\sqrt{(x^2 - 1)}}, \quad . \quad . \quad . \quad (p)$$

in which the negative sign indicates that the primary value of $\operatorname{cosec}^{-1}x$ when x is positive is a decreasing function.

82. To the formulæ found above we add that corresponding to the versed-sine of Art. 72. Let

$$y = \operatorname{vers}^{-1}x, \qquad \text{then} \qquad x = \operatorname{vers} y = 1 - \cos y,$$

$$dx = \sin y\,dy, \qquad dy = \frac{dx}{\sin y}.$$

But $\sin y = \sqrt{(1 - \cos^2 y)} = \sqrt{[1 - (1 - x)^2]} = \sqrt{(2x - x^2)}$; therefore

$$d(\operatorname{vers}^{-1}x) = \frac{dx}{\sqrt{(2x - x^2)}}. \quad . \quad . \quad . \quad . \quad (q)$$

The inverse versed-sine is thus a continuous function for the range from $x = 0$ to $x = 2$, increasing from the value 0 to the value π.

Homogeneous Forms of the Formulæ.

83. In geometrical applications, the independent variable of an inverse circular function usually occurs in the form of the ratio of two lines, in accordance with the definitions of the direct circular functions. Putting x/a in place of x in the foregoing formulæ, we have their homogeneous forms, in which each letter stands for the length of a line, the constant a taking the place of the unit of length. We thus obtain

$$d\left(\sin^{-1}\frac{x}{a}\right) = \frac{dx}{\sqrt{(a^2 - x^2)}};$$

$$d\left(\tan^{-1}\frac{x}{a}\right) = \frac{a\,dx}{a^2 + x^2};$$

$$d\left(\sec^{-1}\frac{x}{a}\right) = \frac{a\,dx}{x\sqrt{(x^2 - a^2)}}.$$

Examples VIII.

1. Derive $d(\sec^{-1}x)$ from the equation $\sec^{-1}x = \cos^{-1}\frac{1}{x}$.

2. Derive $d\left(\cot^{-1}\frac{x}{a}\right)$ from the equation $\cot^{-1}\frac{x}{a} = \tan^{-1}\frac{a}{x}$.

3. $y = \sin^{-1}\frac{x+1}{\sqrt{2}}$. $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{(1-2x-x^2)}}$.

4. $y = \sin^{-1}(2x^2)$. $\qquad \frac{dy}{dx} = \frac{4x}{\sqrt{(1-4x^4)}}$.

5. $y = \sin^{-1}(\cos x)$. $\qquad \frac{dy}{dx} = -1$.

6. $y = \sin(\cos^{-1}x)$. $\qquad \frac{dy}{dx} = -\frac{x}{\sqrt{(1-x^2)}}$.

7. $y = \sin^{-1}(\tan x)$. $\qquad \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{(1-\tan^2 x)}}$.

8. $y = \cos^{-1}(2\cos x)$. $\qquad \frac{dy}{dx} = -\frac{2\sin x}{\sqrt{(1-4\cos^2 x)}}$.

9. $y = x\sin^{-1}x + \sqrt{(1-x^2)}$. $\qquad \frac{dy}{dx} = \sin^{-1}x$.

10. $y = \tan^{-1}e^x$. $\qquad \frac{dy}{dx} = \frac{1}{e^x + e^{-x}}$.

11. $y = (x^2+1)\tan^{-1}x - x$. $\qquad \frac{dy}{dx} = 2x\tan^{-1}x$.

12. $y = a^2\sin^{-1}\frac{x}{a} + x\sqrt{(a^2-x^2)}$. $\qquad \frac{dy}{dx} = 2\sqrt{(a^2-x^2)}$

13. $y = \tan^{-1}\frac{mx}{1-x^2}$. $\qquad \frac{dy}{dx} = \frac{m(1+x^2)}{1+(m^2-2)x^2+x^4}$.

14. $y = \tan^{-1}\frac{x\sqrt{3}}{2+x}$. $\qquad \frac{dy}{dx} = \frac{\sqrt{3}}{2(x^2+x+1)}$.

15. $y = \tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$. $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$.

16. $y = \sec^{-1}\frac{a}{\sqrt{(a^2-x^2)}}$. $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{(a^2-x^2)}}$.

17. $y = \sin^{-1}\dfrac{x}{\sqrt{(x^2+a^2)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{a}{a^2+x^2}$.

18. $y = \sin^{-1}\sqrt{(\sin x)}$. $\qquad \dfrac{dy}{dx} = \frac{1}{2}\sqrt{(1+\operatorname{cosec} x)}$.

19. $y = \sqrt{(1-x^2)}\sin^{-1}x - x$. $\qquad \dfrac{dy}{dx} = -\dfrac{x\sin^{-1}x}{\sqrt{(1-x^2)}}$.

20. $y = \tan^{-1}\dfrac{m+x}{1-mx}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.

21. $y = \cos^{-1}\dfrac{1-x^2}{1+x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}$.

22. $y = \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$. $\qquad \dfrac{dy}{dx} = \frac{1}{2}$.

23. $y = \dfrac{x\sin^{-1}x}{\sqrt{(1-x^2)}} + \log\sqrt{(1-x^2)}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sin^{-1}x}{(1-x^2)}$.

24. $y = (x+a)\tan^{-1}\sqrt{\dfrac{x}{a}} - \sqrt{(ax)}$. $\qquad \dfrac{dy}{dx} = \tan^{-1}\sqrt{\dfrac{x}{a}}$.

25. $y = \exp(\sin^{-1}x)$. $\qquad \dfrac{dy}{dx} = \dfrac{\exp(\sin^{-1}x)}{\sqrt{(1-x^2)}}$.

26. $y = \exp[(1+x^2)\tan^{-1}x]$.

$$\frac{dy}{dx} = (1+2x\tan^{-1}x)\exp[(1+x^2)\tan^{-1}x].$$

27. $y = \dfrac{2\sin^{-1}x}{\sqrt{(1-x^2)}} + \log\dfrac{1-x}{1+x}$. $\qquad \dfrac{dy}{dx} = \dfrac{2x\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$.

28. $y = \sin^{-1}\dfrac{x\tan\alpha}{\sqrt{(a^2-x^2)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^2\tan\alpha}{a^2-x^2}\cdot\dfrac{1}{\sqrt{(a^2-x^2\sec^2\alpha)}}$.

29. $y = \cos^{-1}\sqrt{\left(\dfrac{a^2-x^2}{b^2-x^2}\right)}$. $\qquad \dfrac{dy}{dx} = \dfrac{x\sqrt{(b^2-a^2)}}{(b^2-x^2)\sqrt{(a^2-x^2)}}$.

30. $y = \dfrac{(1-x^2)^{\frac{3}{2}}\sin^{-1}x}{x}$.

$$\frac{dy}{dx} = \frac{1-x^2}{x} - \frac{1+2x^2}{x^2}\sqrt{(1-x^2)}\sin^{-1}x.$$

31. $y = \dfrac{ax-1}{\sqrt{(1+x^2)}}\exp(a\tan^{-1}x)$.

$$\frac{dy}{dx} = \frac{(1+a^2)x}{(1+x^2)^{\frac{3}{2}}}\exp(a\tan^{-1}x).$$

Use logarithmic differentiation.

32. $y = \tan^{-1}[x + \sqrt{(1 - x^2)}]$.

$$\frac{dy}{dx} = \frac{\sqrt{(1 - x^2)} - x}{2\sqrt{(1 - x^2)}[1 + x\sqrt{(1 - x^2)}]}.$$

33. $y = \sin^{-1}\dfrac{b + a \cos x}{a + b \cos x}$. $\qquad \dfrac{dy}{dx} = -\dfrac{\sqrt{(a^2 - b^2)}}{a + b \cos x}$.

34. $y = \sec^{-1}\dfrac{x\sqrt{5}}{2\sqrt{(x^2 + x - 1)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{(x^2 + x - 1)}}$.

35. $y = \tan^{-1}\dfrac{3a^2x - x^3}{a^3 - 3ax^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{3a}{a^2 + x^2}$.

36. $y = \tan^{-1}\dfrac{\sqrt{(b^2 - a^2)} \sin x}{a + b \cos x}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(b^2 - a^2)}}{b + a \cos x}$.

37. $y = \sin^{-1}\dfrac{x\sqrt{(a - b)}}{\sqrt{[a(1 + x^2)]}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(a - b)}}{(1 + x^2)\sqrt{(a + bx^2)}}$.

38. $v = \cos^{-1}\dfrac{x^3 - 2}{x^3}$. $\qquad \dfrac{dy}{dx} = -\dfrac{3}{x\sqrt{(x^3 - 1)}}$.

39. $y = x \exp(\tan^{-1}x)$ $\qquad \dfrac{dy}{dx} = \dfrac{x^2 + x + 1}{x^2 + 1}\exp(\tan^{-1}x)$.

IX.

Recapitulation of Formulæ.

84. We have now obtained the formulæ required in Art. 42 to enable us to differentiate all the functions which can be expressed by combining the elementary functional symbols and the algebraic operations. They are here recapitulated:

$$d(x + y + z + \ldots) = dx + dy + dz + \ldots \quad . \; . \; . \quad (A)$$

$$d(x_1x_2 \ldots x_p) = x_2 \ldots x_p dx_1 + x_1x_3 \ldots x_p dx_2 + \ldots \quad . \quad (B)$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (C)$$

$$d(x^n) = nx^{n-1}dx. \tag{a}$$

$$d(\log_b x) = \frac{dx}{x \log b}. \tag{b}$$

$$d(\log x) = \frac{dx}{x}. \tag{b'}$$

$$d(a^x) = \log a \, . \, a^x \, dx. \tag{c}$$

$$d(e^x) = e^x dx. \tag{c'}$$

$$d(\sin \theta) = \cos \theta \, d\theta. \tag{d}$$

$$d(\cos \theta) = -\sin \theta \, d\theta. \tag{e}$$

$$d(\tan \theta) = \frac{d\theta}{\cos^2 \theta} = \sec^2\theta \, d\theta. \tag{f}$$

$$d(\cot \theta) = -\frac{d\theta}{\sin^2 \theta} = -\operatorname{cosec}^2\theta \, d\theta. \tag{g}$$

$$d(\sec \theta) = \frac{\sin \theta \, d\theta}{\cos^2 \theta} = \sec \theta \tan \theta \, d\theta. \tag{h}$$

$$d(\operatorname{cosec} \theta) = -\frac{\cos \theta \, d\theta}{\sin^2 \theta} = -\operatorname{cosec}\theta \cot \theta \, d\theta. \tag{i}$$

$$d(\operatorname{vers} \theta) = \sin \theta \, d\theta. \tag{j}$$

$$d(\sin^{-1}x) = \frac{dx}{\sqrt{(1 - x^2)}}. \tag{k}$$

$$d(\cos^{-1}x) = -\frac{dx}{\sqrt{(1 - x^2)}}. \tag{l}$$

$$d(\tan^{-1}x) = \frac{dx}{1 + x^2}. \tag{m}$$

$$d(\cot^{-1}x) = -\frac{dx}{1 + x^2}. \tag{n}$$

$$d(\sec^{-1}x) = \frac{dx}{x\sqrt{(x^2 - 1)}}. \tag{o}$$

$$d(\operatorname{cosec}^{-1}x) = -\frac{dx}{x\sqrt{(x^2 - 1)}}. \tag{p}$$

$$d(\text{vers}^{-1}x) = \frac{dx}{\sqrt{(2x - x^2)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (q)$$

Differential of a Function of Two Variables.

85. Formulæ (*a*) . . . (*q*) express the rate of variation in a function due to a known rate of variation in a single variable upon which it depends, by means of the differentials which measure these rates.

This they do in each case by giving the "differential coefficient," or derivative which measures the relative rate.

The formulæ (A), (B) and (C) have hitherto been used in the combination of variables which were themselves functions of a single independent variable. But in them the several variables may be independent, so that they express the rate of a function of several variables in terms of the values of the variables and their rates.

Now the form of these equations is such that the differential of the function is the sum of several terms containing respectively as factors the differentials of the several variables.

These several terms are called *partial differentials*, and their coefficients, *partial differential coefficients.* In contradistinction, the differential of the function of several variables is called *the total differential.*

86. The same thing is obviously true with respect to the form of differentials resulting from the substitution of formulæ (A), (B) and (C) in the other formulæ. For example, substituting from (C) in (*k*), (*m*) and (*o*) respectively, we find

$$d\left(\sin^{-1}\frac{x}{y}\right) = \frac{\dfrac{ydx - xdy}{y^2}}{\sqrt{\left(1 - \dfrac{x^2}{y^2}\right)}} = \frac{ydx - xdy}{y\sqrt{(y^2 - x^2)}}; \quad . \quad (1)$$

$$d\left(\tan^{-1}\frac{x}{y}\right)=\frac{\dfrac{ydx-xdy}{y^2}}{1+\dfrac{x^2}{y^2}}=\frac{ydx-xdy}{x^2+y^2}; \quad . \quad . \quad . \quad . \quad (2)$$

$$d\left(\sec^{-1}\frac{x}{y}\right)=\frac{\dfrac{ydx-xdy}{y^2}}{\dfrac{x}{y}\sqrt{\left(\dfrac{x^2}{y^2}-1\right)}}=\frac{ydx-xdy}{x\sqrt{(x^2-y^2)}}. \quad . \quad (3)$$

Each of these total differentials consists of two parts, one containing dx as a factor and the other dy. The coefficients in these partial differentials are also called the *partial derivatives* of the function of x and y. This principle, namely, that *the total differential is simply the sum of partial differentials*,[*] each of which vanishes when the corresponding variable is made constant, shows that each partial derivative can be found by regarding the function as a function of the corresponding variable alone. Thus the partial derivative which is the coefficient of dx is nothing more than *the derivative with respect to x*.

Accordingly, the coefficient of dy in equation (2) above, or partial derivative for y of $\tan^{-1}\frac{x}{y}$, is simply the derivative of $\tan^{-1}\frac{x}{y}$, or of $\cot^{-1}\frac{y}{x}$, with respect to y. In like manner, in equation (3), the coefficient of dy is the derivative with respect to y of $\cos^{-1}\frac{y}{x}$.

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[*] We do not at present need to show that this is true of every conceivable function capable of differentiation since we are dealing only with the functions expressible by the elementary symbols.

The Derivatives of Implicit Functions.

87. When y is a function of x given in the implicit form, the relation connecting the variables is the result of equating to zero a certain function of x and y. The result of differentiating the relation between x and y is therefore equivalent to equating to zero the total differential of this function. This gives a relation between x, y, dx and dy, by means of which the ratio dy/dx can be expressed in terms of x and y. For example, taking the illustration of an implicit function given in Art. 5, namely

$$ax^2 - 3axy + y^3 - a^3 = 0, \quad . \quad . \quad . \quad . \quad (1)$$

differentiation gives

$$(2ax - 3ay)dx - (3ax - 3y^2)dy = 0,$$

whence

$$\frac{dy}{dx} = \frac{a(2x - 3y)}{3(ax - y^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

88. This equation gives the derivative of the implicit function y, not directly as a function of x, but implicitly so, by virtue of the original equation. It determines the value of the derivative for any known simultaneous values of x and y. Thus, in the illustration above, if we put $y = a$ in equation (1), we obtain $x = 0$ or $x = 3a$. Hence $(0, a)$ and $(3a, a)$ represent simultaneous values of x and y. Denoting the corresponding special values of the derivative by suffixes, we find from equation (2), by substitution,

$$\left.\frac{dy}{dx}\right]_{0,\,a} = 1 \quad \text{and} \quad \left.\frac{dy}{dx}\right]_{3a,\,a} = \frac{1}{2}.$$

Regarding (1) as the rectangular equation of a curve, we have thus determined two points on the curve; and also, for each of them, the value of the gradient at that point.

Examples IX.

Find the total derivatives of the following functions:

1. $u = xye^{x+2y}$. $\quad du = e^{x+2y}[y(1+x)dx + x(1+2y)dy]$.

2. $u = \log\tan\frac{x}{y}$. $\quad du = 2\frac{ydx - xdy}{y^2 \sin 2\frac{x}{y}}$.

3. $u = \log\tan^{-1}\frac{x}{y}$. $\quad du = \frac{ydx - xdy}{(x^2+y^2)\tan^{-1}\frac{x}{y}}$.

4. $u = \frac{\sqrt{x} + \sqrt{y}}{x+y}$.

$$du = \frac{[y - x - 2\sqrt{(xy)}]\sqrt{y}dx + [x - y - 2\sqrt{(xy)}]\sqrt{x}dy}{2\sqrt{(xy)}\,(x+y)^2}.$$

5. $u = \frac{e^x y}{(x^2+y^2)^{\frac{1}{2}}}$. $\quad du = e^x\frac{(x^2+y^2-x)ydx + x^2dy}{(x^2+y^2)^{\frac{3}{2}}}$.

6. $u = \tan^{-1}\frac{x-y}{x+y}$. $\quad du = \frac{ydx - xdy}{x^2+y^2}$.

7. $u = \sqrt{\frac{x^2-y^2}{x^2+y^2}}$. $\quad du = \frac{2xy(ydx - xdy)}{(x^2+y^2)^{\frac{3}{2}}\sqrt{(x^2-y^2)}}$.

8. $u = \log\frac{x+\sqrt{(x^2-y^2)}}{x-\sqrt{(x^2-y^2)}}$. $\quad du = \frac{2(ydx - xdy)}{y\sqrt{(x^2-y^2)}}$.

9. From $x = r\cos\theta$ and $y = r\sin\theta$, deduce

$$(dx)^2 + (dy)^2 = (dr)^2 + r^2(d\theta)^2.$$

10. Given $x = r\cos\theta$ and $y = r\sin\theta$, r and θ being independent variables, prove that

$$dy\sin\theta + dx\cos\theta = dr,$$

and

$$dy\cos\theta - dx\sin\theta = rd\theta.$$

11. Given $x = r\cos\theta$ and $y = r\sin\theta$; eliminate θ and find dr; also eliminate r and find $d\theta$.

$$dr = \frac{xdx + ydy}{\sqrt{(x^2 + y^2)}}, \text{ and } d\theta = \frac{xdy - ydx}{x^2 + y^2}.$$

12. If y is defined as an implicit function by the equation

$$xy^2 + x^2y - 2 = 0,$$

find the values of its derivative corresponding to $x = 1$.

$$\left.\frac{dy}{dx}\right]_{1,1} = -1, \text{ and} \left.\frac{dy}{dx}\right]_{1,-2} = 0.$$

13. Given $x^3(y-1) + y^3(x+1) = 1$; find an expression in terms of x and y for $\frac{dy}{dx}$, and also its numerical values when $y = 2$.

$$\left.\frac{dy}{dx}\right]_{-1,2} = -6, \text{ and } \left.\frac{dy}{dx}\right]_{-7,2} = -\frac{6}{23}.$$

14. Show that the equation

$$xy^3 - 3x^2y + 6y^2 + 2x = 0,$$

is satisfied by (2, 1) and by (0, 0); and find the corresponding values of $\frac{dy}{dx}$.

$$\left.\frac{dy}{dx}\right]_{2,1} = \frac{3}{2}; \quad \left.\frac{dy}{dx}\right]_{0,0} = \infty.$$

15. $\tan^{-1}\frac{x-a}{x+a} - \tan^{-1}\frac{y-a}{y+a} = b.$ $\qquad \frac{dy}{dx} = \frac{y^2 + a^2}{x^2 + a^2}.$

16. $y = 1 + xe^y.$ $\qquad \frac{dy}{dx} = \frac{e^y}{2-y}.$

17. $(x-y)y^n = x + y.$ $\qquad \frac{dy}{dx} = \frac{2y^2}{2xy - n(x^2 - y^2)}.$

18. $(x^2 + y^2)^2 = a^2x^2 - b^2y^2.$ $\qquad \frac{dy}{dx} = \frac{[a^2 - 2(x^2 + y^2)]x}{[b^2 + 2(x^2 + y^2)]y}.$

19. $ye^{ny} = ax^m.$ $\qquad \frac{dy}{dx} = \frac{my}{x(1 + ny)}.$

20. $y^3 - 3y\sin^{-1}x + x^3 = 0.$ $\qquad \frac{dy}{dx} = 3y\frac{y - x^2(1 - x^2)^{\frac{1}{2}}}{(2y^3 - x^3)(1 - x^2)^{\frac{1}{2}}}.$

21. $y \sin nx - ae^{nx+y} = 0.$ $\qquad \dfrac{dy}{dx} = \dfrac{ny}{1-y}(1 - \cot nx).$

22. $y \tan^{-1}x - y^2 + x^2 = 0.$ $\qquad \dfrac{dy}{dx} = \dfrac{y(y + 2x + 2x^3)}{(1 + x^2)(y^2 + x^2)}.$

Miscellaneous Examples.

1. $y = \dfrac{x}{\sqrt{(1+x)}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{x+2}{2(1+x)^{\frac{3}{2}}}.$

2. $y = \sqrt{\dfrac{a^2 - x^2}{b^2 - x^2}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{(a^2 - b^2)x}{(a^2 - x^2)^{\frac{1}{2}}(b^2 - x^2)^{\frac{3}{2}}}.$

3. $y = \dfrac{\sqrt{(a+x)}}{\sqrt{a} + \sqrt{x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a}(\sqrt{x} - \sqrt{a})}{2\sqrt{x}\sqrt{(a+x)}(\sqrt{a} + \sqrt{x})^2}.$

4. $y = (\sqrt{x} - 2\sqrt{a})\sqrt{(\sqrt{a} + \sqrt{x})}.$ $\qquad \dfrac{dy}{dx} = \dfrac{3}{4\sqrt{(\sqrt{a} + \sqrt{x})}}.$

5. $y = \dfrac{(x-1)(e^x + 1)e^x}{e^x - 1}.$ $\qquad \dfrac{dy}{dx} = \dfrac{e^x(xe^{2x} - 2xe^x + 2e^x - x)}{(e^x - 1)^2}.$

6. $y = \log \dfrac{(1 + x^2)^{\frac{1}{4}}}{(1 + x)^{\frac{1}{2}}} + \frac{1}{2}\tan^{-1}x.$ $\qquad \dfrac{dy}{dx} = \dfrac{x}{(1+x)(1+x^2)}.$

7. $y = \log\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}} - \frac{1}{2}\tan^{-1}x.$ $\qquad \dfrac{dy}{dx} = \dfrac{x^2}{1 - x^4}.$

8. $y = \log\,[x + \sqrt{(x^2 - a^2)}] + \sec^{-1}\dfrac{x}{a}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x}\sqrt{\left(\dfrac{x+a}{x-a}\right)}.$

9. $y = \dfrac{x^3 - x}{(x^2 + 1)^2} + \tan^{-1}x.$ $\qquad \dfrac{dy}{dx} = \dfrac{8x^2}{(x^2+1)^3}.$

10. $y = a \log \dfrac{a + \sqrt{(a^2 - x^2)}}{x} - \sqrt{(a^2 - x^2)}.$

$$\frac{dy}{dx} = -\frac{\sqrt{(a^2 - x^2)}}{x}.$$

11. $y = \dfrac{(1 + 3x + 3x^2)^{\frac{1}{3}}}{x}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{(1+x)^2}{x^2(1 + 3x + 3x^2)^{\frac{2}{3}}}.$

12. $y = a \log \dfrac{a + \sqrt{(a^2 - x^2)}}{x} - \sqrt{(a^2 - x^2)}.$

$$\frac{dy}{dx} = -\frac{\sqrt{(a^2 - x^2)}}{x}.$$

13. $y = \log \sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sin x}$.

14. $y = \tan^{-1}\left[\sqrt{\dfrac{a-b}{a+b}} \cdot \tan\dfrac{x}{2}\right]$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(a^2 - b^2)}}{2(a + b\cos x)}$.

15. $y = \sec^{-1}\dfrac{1}{2x^2 - 1}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2}{\sqrt{(1 - x^2)}}$.

16. $y = \cos^{-1}\dfrac{x^{2n} - 1}{x^{2n} + 1}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2nx^{n-1}}{x^{2n} + 1}$.

17. $y = a\cos^{-1}\dfrac{a - x}{b} - \sqrt{[b^2 - (a - x)^2]}$.

$$\frac{dy}{dx} = \frac{x}{\sqrt{[b^2 - (a - x)^2]}}.$$

18. $y = \cos^{-1}x - 2\sqrt{\dfrac{1 - x}{1 + x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(1 - x)}}{(1 + x)^{\frac{3}{2}}}$.

19. $y = x^{y+x}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^{y+x}(x^{y+x-1} + \log x + 1)}{1 - x^{y+x}\log x}$.

20. $y = x^{x^n}$. $\qquad \dfrac{dy}{dx} = (n\log x + 1)x^{x^n}x^{n-1}$.

21. $y = x^{x^x}$. $\qquad \dfrac{dy}{dx} = x^{x^x}x^x\left[(\log x)^2 + \log x + \dfrac{1}{x}\right]$.

22. $y = \exp(x\tan\beta)\sin(x + \beta)$. $\qquad \dfrac{dy}{dx} = \dfrac{\exp(x\tan\beta)\cos x}{\cos\beta}$.

23. $y = \log[x^2 + \sqrt{(1 + x^4)}]$. $\qquad \dfrac{dy}{dx} = \dfrac{2x}{\sqrt{(1 + x^4)}}$.

24. $y = \tan^{-1}\dfrac{\sqrt{(a + bx)}}{\sqrt{(b - a)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(b - a)}}{2(1 + x)\sqrt{(a + bx)}}$.

25. $y = \log\tan\dfrac{x}{2} - \dfrac{\cos x}{\sin^2 x}$. $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sin^3 x}$.

26. $y = \log\dfrac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + 2\tan^{-1}\dfrac{x\sqrt{2}}{1 - x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{4\sqrt{2}}{1 + x^4}$.

27. $y = \dfrac{1}{(1 + x)^4}\left(\dfrac{1}{x} + \dfrac{125}{12} + \dfrac{65x}{3} + \dfrac{35x^2}{2} + 5x^3\right) + 5\log\dfrac{x}{1 + x}$.

$$\frac{dy}{dx} = -\frac{1}{x^2(1 + x)^5}.$$

28. $y = \log\frac{1+x}{1-x} + \frac{1}{2}\log\frac{1+x+x^2}{1-x+x^2} + \sqrt{3}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}.$

$$\frac{dy}{dx} = \frac{6}{1-x^6}.$$

29. $y = (1+x^2)^{\frac{m}{2}}\sin(m\tan^{-1}x).$

$$\frac{dy}{ax} = m(1+x^2)^{\frac{m-1}{2}}\cos[(m-1)\tan^{-1}x].$$

30. $y = \log\frac{(x-1)^2}{x^2+x+1} - 2\sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$ $\qquad \frac{dy}{dx} = \frac{6}{x^3-1}.$

31. If $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, show that

$$\frac{dy}{dx} = 1 - y^2.$$

32. Given $u = xz + a\sin z + az\cos z$, and $x = a - a\cos z$; prove that

$$\frac{du}{dx} = \left(\frac{2a-x}{x}\right)^{\frac{1}{2}}.$$

CHAPTER III.

SUCCESSIVE DERIVATIVES.

X.

Velocity and Acceleration.

89. WE have in Art. 17 employed the velocity of a moving point to illustrate the rate of a variable, the variable x being represented by the distance of the moving point from a fixed origin in the line of motion. If we now represent this velocity by v, we have

$$v = \frac{dx}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When this velocity is variable its rate of variation is *the rate of the rate of x*. Since dt is constant, we have, by differentiating equation (1),

$$dv = \frac{d(dx)}{dt}, \qquad \text{whence} \qquad \frac{dv}{dt} = \frac{d(dx)}{(dt)^2}. \quad . \quad . \quad (2)$$

The rate of the velocity is called the *acceleration* of the moving point, and may be denoted by the single letter α. In equation (2), $d(dx)$ is generally written in the abbreviated form d^2x, which may be read "*d*-second" x; also the marks of parenthesis are omitted in the denominator. Thus we write

$$\alpha = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

90. In equations (1) and (3), x, the *space described*, is a definite function of t; v, the velocity, is the derivative of the space with respect to t; while α, the acceleration, is the derivative of v with respect to t, and is called *the second derivative* of x with respect to t.

Just as a positive value of the first derivative v indicates *algebraic* increase of x, so a positive value of the second derivative α indicates *algebraic* increase of v. The term acceleration is of course derived from the case when both the velocity and its rate are positive, so that the moving point is *hastened.* A negative acceleration is a retardation of a positive velocity, but an algebraic increase of a negative one.

91. For example, suppose it to be known that the space described in the time t by a freely falling body varies as the square of the time, so that it may be represented by

$$x = \tfrac{1}{2}gt^2,$$

where g is a positive constant. From this we derive

$$v = \frac{dx}{dt} = gt,$$

and

$$\alpha = \frac{dv}{dt} = \frac{d^2x}{dt^2} = g.$$

In this case, therefore, the acceleration is constant and positive. Accordingly, the velocity which is positive for positive values of t, is increasing. At the instant when $t = 0$ we have $x = 0$ and $v = 0$. Supposing the body to be already in motion before that instant, we see that v is negative for negative values of t, that is, the body was moving in the opposite (or upward) direction, and then the positive acceleration implied a decrease in the negative velocity.

92. The time, space, velocity and acceleration may be regarded as four variables connected by the two general differential relations

$$v = \frac{dx}{dt}, \qquad \alpha = \frac{dv}{dt}.$$

Therefore, when one other relation between them is given, three of the four variables become definite functions of a single independent variable, which may be any one of the four. The problem of so expressing them under different forms of the additional datum relation (which makes the motion definite) is the application of the Calculus to the subject of *rectilinear motion*, and for the most part requires the inverse process of Integration.

We may here, however, notice another general differential relation found by eliminating dt from the two given above, Thus

$$\alpha = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}; \quad . \quad . \quad . \quad (1)$$

that is to say, *the acceleration is always equal to the space-derivative of one-half the squared velocity.*

By means of this we can, when v is given in terms of x, express α in terms of x. For example, if we are given $v = n\sqrt{(x^2 - a^2)}$, we thus find $\alpha = n^2x$.

Component Velocities and Accelerations.

93. When a point moves in a plane curve its motion is most conveniently discussed by means of points geometrically connected with it which have rectilinear motion. Referring the path of the point P to rectangular coordinates as in Fig. 18, these points are the projections R and S of P upon the axes. Their velocities (which are the rates of x and y respectively) are called ***the component velocities*** of P. The actual velocity of P is the rate of s, the space described measured along the curve from some fixed point of it, as A in the figure. Denoting it by v, and the component velocities by v_x and v_y, we have

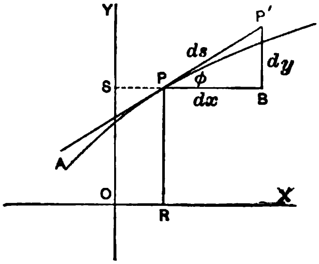

FIG. 18.

$$v = \frac{ds}{dt}, \qquad v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}.$$

94. In the differential triangle, constructed as in Art. 37, PP' represents ds and $P'PB$ is ϕ, the angle of slope. Then, since the triangle is right-angled, we have

$$dx = \cos\phi\, ds, \qquad dy = \sin\phi\, ds$$

and

$$ds^2 = dx^2 + dy^2.$$

Dividing by dt and dt^2 respectively, we find

$$v_x = v\cos\phi, \qquad v_y = v\sin\phi$$

and

$$v^2 = v_x^2 + v_y^2.$$

These equations serve to determine the actual velocity, v, and the slope of the curve, when the component velocities are given.

In these equations, ϕ is the angle of inclination of the actual motion of P, thus distinguishing between the two values of ϕ which in Art. 38 correspond to the same gradient.

95. The accelerations of R and S may be denoted by α_x and α_y; thus,

$$\alpha_x = \frac{d^2x}{dt^2}, \qquad \alpha_y = \frac{d^2y}{dt^2}.$$

These are called *the component accelerations* of the point P. It is evident that their values determine not only the acceleration of the point P in its path, but the curvature of this path.

Examples X.

1. The space in feet described in the time t by a point moving in a straight line is expressed by the formula

$$x = 48t - 16t^2;$$

find the acceleration, and the velocity at the end of $2\frac{1}{2}$ seconds; also find the value of t for which $v = 0$.

$\alpha = -32$; $v = 0$, when $t = 1\frac{1}{2}$.

2. If the space described in t seconds be expressed by the formula

$$x = 10 \log \frac{4}{4 + t},$$

find the velocity and acceleration at the end of 1 second and at the end of 16 seconds. When $t = 1$, $v = -2$ and $\alpha = \frac{2}{5}$.

3. If a point moves in a fixed path so that

$$s = \sqrt{t},$$

show that the acceleration is negative and proportional to the cube of the velocity. Find the value of the acceleration at the end of one second and at the end of nine seconds. $-\frac{1}{4}$; $-\frac{1}{108}$.

4. If a point move in a straight line so that

$$x = a \cos \tfrac{1}{2}\pi t,$$

show that

$$\alpha = -\tfrac{1}{4}\pi^2 x.$$

5. If

$$x = ae^t + be^{-t},$$

prove that

$$\alpha = x.$$

6. If a point move so that $v = \sqrt{(2gx)}$, determine the acceleration. *Use equation* (1), *Art.* 92. $\alpha = g.$

7. If a point move so that we have

$$v^2 = c - \mu \log x,$$

determine the acceleration. $\alpha = -\dfrac{\mu}{2x}.$

8. If a point move so that we have

$$v^2 = c + \frac{2\mu}{\sqrt{(x^2 + b^2)}},$$

determine the acceleration. $\alpha = \dfrac{\mu x}{(x^2 + b^2)^{\frac{3}{2}}}.$

9. The velocity of a point is inversely proportional to the square of its distance from a fixed point of the straight line in which it moves, the velocity being 2 feet per second when the distance is 6 inches; determine the acceleration at the distance s feet from the fixed point.

$-\dfrac{1}{2s^5}$ feet.

10. The velocity of a point moving in a straight line is m times its distance from a fixed point at the perpendicular distance a from the straight line; determine the acceleration at the distance x from the foot of the perpendicular $\alpha = m^2 x.$

11. The relation between x and t being expressed by

$$t\sqrt{\frac{2\mu}{a}} = \sqrt{(ax - x^2)} - \tfrac{1}{2}a \operatorname{vers}^{-1}\frac{2x}{a};$$

find the acceleration in terms of x. $\alpha = -\dfrac{\mu}{x^2}.$

12. If $v^2 = A + \frac{B}{x}$, show that the acceleration varies inversely as the square of the distance from a fixed point in the line of motion.

13. If $v^2 = A + Bx + Cx^2$, show that the acceleration varies as the distance from a fixed point in the line of motion.

14. In "tram motion" each end of a rod AB is constrained to move in one of two grooves crossing each other at right angles at O. If the velocity of one end is proportional to the distance of the other end from O, prove that its acceleration is proportional to its own distance from O.

15. A point referred to rectangular coordinate axes moves so that

$$x = a \cos t + b, \qquad y = a \sin t + c;$$

show that the velocity is constant and that ϕ uniformly increases. Find also the equation of the path described.

16. A projectile moves in the parabola whose equation is

$$y = x \tan \alpha - \frac{g}{2V^2 \cos^2 \alpha} x^2$$

(the axis of y being vertical) with the uniform horizontal velocity

$$v_x = V \cos \alpha;$$

find the velocity in the curve, and the vertical acceleration.

$$v = \sqrt{(V^2 - 2gy)}; \quad \alpha_y = -g.$$

17. A point moves in the curve, whose equation is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

in such a manner that v_x is constant and equal to k; find the acceleration in the direction of the axis of y.

$$\alpha_y = \frac{a^{\frac{2}{3}} k^2}{3 x^{\frac{4}{3}} y^{\frac{1}{3}}}.$$

18. A point moves in the hyperbola

$$y^2 = p^2 x^2 + q^2$$

in such a manner that v_x has the constant value c; prove that

$$v_y^2 = p^2c^2 - \frac{p^2c^2q^2}{y^2},$$

and thence derive α_y. $\alpha_y = \frac{p^2c^2q^2}{y^3}$.

19. A point describes the conic section

$$y^2 = 2mx + nx^2,$$

v_x having the constant value c; determine the value of α_y in terms of y.

$$\alpha_y = -\frac{m^2c^2}{y^3}.$$

XI.

Successive Derivatives of a Function.

96. The derivative of $f(x)$ is another function of x, which we have denoted by $f'(x)$; if we take the derivative of the latter, we obtain still another function of x, which is called the second derivative of the original function $f(x)$, and is denoted by $f''(x)$. Thus if

$$f(x) = x^3, \qquad f'(x) = 3x^2 \qquad \text{and} \qquad f''(x) = 6x.$$

Similarly the derivative of $f''(x)$ is denoted by $f'''(x)$, and is called the third derivative of $f(x)$; etc. When one of these successive derivatives has a constant value, the next and all succeeding derivatives evidently vanish. Thus, in the above example, $f'''(x) = 6$, consequently, in this case, $f^{\text{IV}}(x)$ and all higher derivatives vanish.

97. When the function is denoted by the single letter y, we have seen in Art. 35 that $\frac{d}{dx}$ may be taken as the symbol of the operation of taking the derivative. The single letter D is often used for the same purpose, and an exponent is applied to the symbol to denote repetitions of the operation; thus $D(Dy)$ or D^2y is the second derivative of y, and D^ny is the nth derivative. In like manner the higher derivatives may be denoted by

$$\left(\frac{d}{dx}\right)^2 y, \quad \left(\frac{d}{dx}\right)^3 y, \ldots \left(\frac{d}{dx}\right)^n y$$

in which the independent variable is directly expressed. These last symbols are more usually written in the abbreviated forms

$$\frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \ldots \frac{d^ny}{dx^n},$$

although the former symbols are the more accurate, because the operation to be performed n times is that of *differentiating and removing the factor dx after each differentiation.*[*]

Geometrical Meaning of the Second Derivative.

98. When the graph of the function $y = f(x)$ is drawn, we have seen that

$$\frac{dy}{dx} = f'(x) = \tan \phi,$$

ϕ being the inclination of the curve to the axis of x; hence

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d(\tan \phi)}{dx}.$$

---

[*] It is to be noticed that, on this account, it is immaterial whether dx is constant or variable. But when dx can be assumed constant (like dt in Art. 89), we may suppose all the differentiations performed first, and $(dx)^n$ removed by division afterward.

If now we suppose a point to describe the curve in such a way that the rate of x is constant and positive, the value of the second derivative gives the rate at which $\tan \phi$, the gradient of the curve, varies. In Fig. 19 are shown several curves having a common tangent, MN, and a common point of contact at C, so that the value of the functions represented and also of their first derivatives are equal at C. But it is obvious that in the curve AB the gradient is increasing more rapidly than it is in the curve $A'B'$, which lies nearer the tangent and is therefore said to have less *curvature*. Thus the value of the second derivative at C is greater for the curve AB than for the curve $A'B'$. Again, for the tangent itself, which represents a linear function, the value of $\frac{d^2y}{dx^2}$ is zero; while for the curve $A''B''$ (in which the gradient is decreasing as x increases) its value is negative.

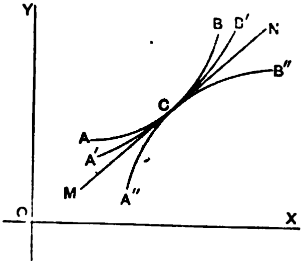

FIG. 19.

Accordingly, when the second derivative is *positive*, the curve lies like AB *above* the tangent line and is *concave as viewed from above;* and, when the second derivative is *negative*, it lies *below* the tangent and is *convex* as viewed from above.

99. When a curve crosses the tangent line at the point of contact, in which case that point separates a convex from a concave portion of the curve, as in Fig. 20, the point is called a *point of inflexion* or of *contrary flexure*. Suppose the point of contact, *carrying the tangent with it*, to move in the positive direction along the curve. As it passes through a point of inflexion ϕ changes from a state of decreasing to a state of increasing, as in the figure, or *vice versa*.

The tangent at the point of inflexion is called a *stationary tangent*, because after turning in one direction it stops and then begins to turn in the opposite direction. The value of the second derivative therefore changes sign as x passes through a certain value; hence, if it is a continuous function of x, it must take the value zero. Hence, to find the abscissa of a point of inflexion, we put the second derivative equal to zero, and if the equation so formed has a root for which the function is real we must then ascertain whether the second derivative changes sign.

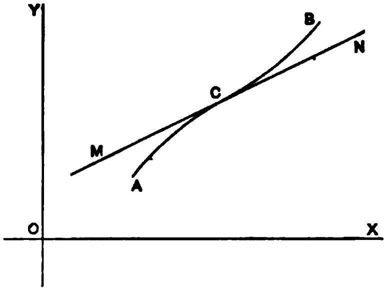

Fig. 20.

For example, the equation $y = x^3$ gives $\frac{d^2y}{dx^2} = 6x$, which vanishes when $x = 0$; the curve has a point of inflexion at the origin because $6x$ changes sign as x passes through zero. Again, $y = x^4$ gives $\frac{d^2y}{dx^2} = 12x^2$, which also vanishes when $x = 0$; but, since it does not change sign, there is, in this case, no point of inflexion.

Higher Derivatives of Implicit Functions.

100. When y is given as an implicit function of x, the higher derivatives, like the first derivative (Art. 87), can in general be found only in terms of x and y; hence the numerical values of these derivatives can be determined only for known simultaneous values of x and y. The following examples will serve to illustrate the method of finding such derivatives.

Given

$$\log(x+y) = x - y; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we obtain, by differentiating and reducing,

$$(x+y+1)dy + (1-x-y)dx = 0; \quad . \quad . \quad (2)$$

whence

$$\frac{dy}{dx} = \frac{x+y-1}{x+y+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Differentiating, and dividing by dx,

$$\frac{d^2y}{dx^2} = \frac{(x+y+1)\left(1+\frac{dy}{dx}\right) - (x+y-1)\left(1+\frac{dy}{dx}\right)}{(x+y+1)^2};$$

substituting the value of $\frac{dy}{dx}$, we obtain

$$\frac{d^2y}{dx^2} = \frac{4(x+y)}{(x+y+1)^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In like manner, the third derivative may be found.

Simultaneous values of x and y are readily found in this case. Thus, if we put $x+y=1$, we have $x-y=0$, whence $x=\frac{1}{2}$ and $y=\frac{1}{2}$; by substituting these values in equations (3) and (4) we obtain

$$\left.\frac{dy}{dx}\right]_{\frac{1}{2},\,\frac{1}{2}} = 0 \qquad \text{and} \qquad \left.\frac{d^2y}{dx^2}\right]_{\frac{1}{2},\,\frac{1}{2}} = \frac{1}{2}.$$

These results show that the curve represented by equation (1) passes through the point $(\frac{1}{2}, \frac{1}{2})$, is there parallel to the axis of x and lies above the tangent line.

Examples XI.

1. If $f(x) = \frac{1+x}{1-x}$, find $f^{\text{v}}(x)$. $\quad f^{\text{v}}(x) = \frac{240}{(1-x)^6}$.

2. If $f(x) = \frac{a}{x^n}$, find $f'''(x)$. $\quad f'''(x) = -\frac{n(n+1)(n+2)a}{x^{n+3}}$.

3. If y is a function of x of the form

$$Ax^n + Bx^{n-1} + \ldots + Mx + N,$$

prove that

$$\frac{d^ny}{dx^n} = n!\ A.$$

4. If $f(x) = b^{ax}$, find $f^{\text{v}}(x)$. $\quad f^{\text{v}}(x) = a^5(\log b)^5 b^{ax}$.

5. If $f(x) = x^3 \log(mx)$, find $f^{\text{iv}}(x)$. $\quad f^{\text{iv}}(x) = \frac{6}{x}$.

6. If $f(x) = \log \sin x$, find $f'''(x)$. $\quad f'''(x) = \frac{2 \cos x}{\sin^3 x}$.

7. If $f(x) = \sec x$, find $f''(x)$ and $f'''(x)$.

$f''(x) = 2 \sec^3 x - \sec x$; $f'''(x) = \sec x \tan x(6 \sec^2 x - 1)$.

8. If $f(x) = \tan x$, find $f'''(x)$ and $f^{\text{iv}}(x)$.

$f'''(x) = 6 \sec^4 x - 4 \sec^2 x$; $f^{\text{iv}}(x) = 8 \tan x \sec^2 x(3 \sec^2 x - 1)$.

9. If $f(x) = x^x$, find $f''(x)$. $\quad f''(x) = x^x(1 + \log x)^2 + x^{x-1}$.

10. If $y = e^{\frac{1}{x}}$, find $\frac{d^3y}{dx^3}$. $\quad \frac{d^3y}{dx^3} = -\frac{1}{x^6}(1 + 6x + 6x^2)e^{\frac{1}{x}}$.

11. If $y = e^{-x^2}$ find D^3y. $\quad D^3y = 4x(3 - 2x^2)e^{-x^2}$.

12. If $y = \log(e^x + e^{-x})$, find $\frac{d^3y}{dx^3}$. $\quad \frac{d^3y}{dx^3} = -8\frac{e^x - e^{-x}}{(e^x + e^{-x})^3}$.

13. If $y = \frac{1}{e^x - 1}$, find $\frac{d^2y}{dx^2}$ and $\frac{d^4y}{dx^4}$.

$$\frac{d^2y}{dx^2} = \frac{e^{2x} + e^x}{(e^x - 1)^3}; \quad \frac{d^4y}{dx^4} = \frac{e^x + 11e^{2x} + 11e^{3x} + e^{4x}}{(e^x - 1)^5}.$$

14. If $v = \sin^{-1}x$, find D^4y. $\quad D^4y = \frac{9x + 6x^3}{(1 - x^2)^{\frac{7}{2}}}$.

15. If $y = e^{\sin x}$, find D^3y.

$$D^3y = -e^{\sin x}\cos x \sin x\,(\sin x + 3).$$

16. If $y = \dfrac{x}{1 + \log x}$, find $\dfrac{d^2y}{dx^2}$. $\qquad \dfrac{d^2y}{dx^2} = \dfrac{1 - \log x}{x(1 + \log x)^3}$.

17. If $y = (\cos^{-1}x)^2$, show that the following relation exists between its derivatives:

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2.$$

18. If $y = a \cos \log x + b \log \sin x$, show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

19. If $xy = ae^x + be^{-x}$, show that

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0.$$

20. If $y = Ae^x \sin(x + \alpha)$, show that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

21. Find the value of $\dfrac{d^3}{dt^3}(\sin\theta)$, θ being a known function of t.

$$\frac{d^3}{dt^3}(\sin\theta) = -\cos\theta\left(\frac{d\theta}{dt}\right)^3 - 3\sin\theta\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + \cos\theta\frac{d^3\theta}{dt^3}.$$

22. If y is a function of x, and if $u = y^2 \log y$; find $\dfrac{d^2u}{dx^2}$.

$$\frac{d^2u}{dx^2} = (2\log y + 3)\left(\frac{dy}{dx}\right)^2 + y(2\log y + 1)\frac{d^2y}{dx^2}.$$

23. If $u = \dfrac{x}{y}$, y being a function of x, find $\dfrac{d^2u}{dx^2}$.

$$\frac{d^2u}{dx^2} = -\frac{2}{y^2}\frac{dy}{dx} + \frac{2x}{y^3}\left(\frac{dy}{dx}\right)^2 - \frac{x}{y^2}\frac{d^2y}{dx^2}.$$

24. Distinguish the concave from the convex portions of the curves $y = \sec x$, $y = \tan x$ and $y = \sin x$.

25. Find the point of inflexion of the curve

$$y = 2x^3 - 3x^2 - 12x + 6. \qquad (\tfrac{1}{2}, -\tfrac{1}{2}).$$

26. What portion of the curve

$$y = x^4 - 2x^3 - 12x^2 + 11x + 24$$

is convex? Between $(2, -2)$ and $(-1, 4)$.

27. Show geometrically that at a point of a curve *where the gradient is positive* $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^2x}{dy^2}$ have opposite signs.

Consider the position of the curve relatively to the tangent line, as in Art. 98.

28. Determine $\dfrac{d^2x}{dy^2}$ when $y = f(x)$. $\qquad \dfrac{d^2x}{dy^2} = -\dfrac{f''(x)}{[f'(x)]^3}$.

29. Find the value of $\dfrac{d^2y}{dx^2}$ for the curve and points considered in Arts. 87 and 88.

$$\left.\frac{d^2y}{dx^2}\right]_{0,\,a} = -\frac{2}{3a}; \qquad \left.\frac{d^2y}{dx^2}\right]_{3a,\,a} = -\frac{1}{12a}.$$

30. Show for the same curve that it lies above the tangent at the point $(-a, 0)$ and below it the point $(a, 0)$, and that the curvature is the same at these two points.

31. If $y - 1 - xe^y = 0$, find $\dfrac{d^2y}{dx^2}$. $\qquad \dfrac{d^2y}{dx^2} = \dfrac{3-y}{(2-y)^3} \cdot e^{2y}$.

32. If $y = \tan(x + y)$, find $\dfrac{d^3y}{dx^3}$. $\qquad \dfrac{d^3y}{dx^3} = -\dfrac{2(5 + 8y^2 + 3y^4)}{y^8}$.

33. If $y^3 + y = x^2$, find $\dfrac{d^3y}{dx^3}$. $\qquad \dfrac{d^3y}{dx^3} = -\dfrac{24x}{(1 + 2y)^5}$.

34. Given $e^x + x = e^y + y$, find $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^2x}{dy^2}$, and show that they satisfy the relation found in Ex. 28.

$$\frac{d^2y}{dx^2} = \frac{(e^{x+y} - 1)(x - y)}{(e^y + 1)^3}.$$

35. Given $e^y + xy - e = 0$, find $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = y\frac{(2-y)e^y + 2x}{(e^y + x)^3}.$$

36. Given $y^3 - 3axy + x^3 = 0$, find $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = -\frac{2a^3xy}{(y^2 - ax)^3}.$$

XII.

Expressions for the nth Derivative.

101. When the derivative of a function can be put in a form similar to that of the function itself, a general expression for the derivative of any order may be written. Thus, because $De^x = e^x$ we have obviously $D^n e^x = e^x$, where n is any positive integer. So also, since the derivative of sin x, which is cos x, may be written sin $(x + \frac{1}{2}\pi)$, we have

$$D^n \sin(x + \alpha) = \sin\left(x + \alpha + \frac{n\pi}{2}\right),$$

which includes, as a particular case,

$$D^n \cos x = \cos\left(x + \frac{n\pi}{2}\right).$$

102. Again, taking the derivative of x^n r times in succession, we have

$$\frac{d^r(x^n)}{dx^r} = n(n-1)\ .\ .\ .\ (n-r+1)x^{n-r},$$

which vanishes if n is a positive integer when $r \geqq n+1$, but never vanishes if n is fractional or negative. In particular, if n is a negative integer, putting $n = -m$, the equation may be written

$$\frac{d^r(x^{-m})}{dx^r} = (-1)^r m(m+1)\ldots(m+r-1)x^{-(m+r)}.$$

When $m = 1$, this becomes

$$\frac{d^r}{dx^r}\left(\frac{1}{x}\right) = (-1)^r \frac{r!}{x^{r+1}}.$$

Since $D \log x = \dfrac{1}{x}$, it follows that

$$D^r \log x = (-1)^{r-1}\frac{(r-1)!}{x^r}.$$

103. The derivative of a function does not necessarily bear any resemblance in form to the function itself; but, in some cases, a more or less obvious device suffices to reduce it to the required form, so as to enable us to express the nth derivative. For example, let

$$y = e^{ax}\cos(bx), \quad \ldots\ldots \quad (1)$$

then

$$dy = e^{ax}[a\cos(bx) - b\sin(bx)]dx.$$

Employing an auxiliary constant α determined by

$$b = a\tan\alpha, \quad \ldots\ldots \quad (2)$$

we have

$$\frac{dy}{dx} = \frac{a}{\cos\alpha}e^{ax}[\cos(bx)\cos\alpha - \sin(bx)\sin\alpha],$$

or

$$Dy = a\sec\alpha\, e^{ax}\cos(bx+\alpha). \quad \ldots \quad (3)$$

Therefore the operation of D upon this function is to multiply by the constant factor $a \sec\alpha$ and to add the constant α

to the angle involved. Hence, repeating the operation, we have

$$D^n y = a^n \sec^n \alpha \, e^{ax} \cos(bx + n\alpha);$$

or, since, by equation (2), $a \sec \alpha = \sqrt{(a^2 + b^2)}$,

$$D^n[e^{ax} \cos(bx)] = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + n \tan^{-1} \frac{b}{a}\right). \quad (4)$$

This formula represents a series of functions of which it will be noticed that the original function is the member corresponding to $n = 0$.

104. The successive derivatives of $y = \cot^{-1} \frac{x}{a}$, though bearing no resemblance in form to the original function, yet follow a law which is detected by expressing them in terms of y. Thus, let

$$y = \cot^{-1} \frac{x}{a}, \qquad \text{then} \qquad x = a \cot y; \quad . \quad . \quad (1)$$

differentiating, $dx = -a \operatorname{cosec}^2 y \, dy$; whence

$$\frac{dy}{dx} = -\frac{\sin^2 y}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Taking the derivative, we have

$$\frac{d^2y}{dx^2} = -\frac{2}{a} \sin y \cos y \frac{dy}{dx} = \frac{1}{a^2} \sin 2y \sin^2 y. \quad . \quad . \quad (3)$$

Again,

$$\frac{d^3y}{dx^3} = \frac{2}{a^2} \sin y (\sin 2y \cos y + \cos 2y \sin y) \frac{dy}{dx},$$

and, substituting from equation (2),

$$\frac{d^3y}{dx^3} = -\frac{1 \,.\, 2}{a^3} \sin 3y \sin^3 y. \quad . \quad . \quad . \quad . \quad (4)$$

In like manner we obtain

$$\frac{d^4y}{dx^4} = \frac{1 \,.\, 2 \,.\, 3}{a^4} \sin 4y \sin^4 y,$$

and, in general,

$$\frac{d^ny}{dx^n} = (-1)^n \frac{(n-1)!}{a^n} \sin ny \sin^n y.$$

Finally, since from equation (1) $\sin y = \dfrac{a}{\sqrt{(a^2+x^2)}}$,

$$\left(\frac{d}{dx}\right)^n \cot^{-1}\frac{x}{a} = (-1)^n \frac{(n-1)!}{(a^2+x^2)^{\frac{n}{2}}} \sin\left[n \cot^{-1}\frac{x}{a}\right]. \quad (5)$$

The *nth* derivative of $\tan^{-1}\dfrac{x}{a}$ is the same expression with its sign changed.

Leibnitz' Theorem.

105. By means of the following theorem, which is due to Leibnitz, the higher derivatives of the product of two functions is expressed in terms of the successive derivatives of the given functions. Let u and v be functions of x; then $d(uv) = udv + vdu$, and using D to denote the derivative with respect to x,

$$D(uv) = u \,.\, Dv + Du \,.\, v. \quad . \quad . \quad . \quad . \quad (1)$$

Thus the derivative of the product is the sum of two terms of which the first is the value it would have if u were constant, and the second the value it would have if v were constant.

Applying this principle in taking the derivative of the second member of equation (1), we derive

$$\begin{aligned} D^2(uv) = u\,.\,D^2v &+ Du\,.\,Dv \\ &+ Du\,.\,Dv + D^2u\,.\,v, \end{aligned}$$

in which the first line is the result of treating the u-factor of each term as a constant, and the second that of treating the v-factor as a constant. Thus we have

$$D^2(uv) = uD^2v + 2DuDv + D^2u\,.\,v, \quad . \quad . \quad . \quad (2)$$

in which the coefficients are those of the expansion of $(a+b)^2$. Again, the application of the same principle to equation (2) gives

$$\begin{aligned} D^3(uv) = uD^3v &+ 2DuD^2v + D^2uDv \\ &+ DuD^2v + 2D^2uDv + D^3u\,.\,v, \end{aligned}$$

or

$$D^3(uv) = uD^3v + 3DuD^2v + 3D^2uDv + D^3u\,.\,v, \quad . \quad (3)$$

in which the numerical coefficients are those of the expansion of $(a+b)^3$.

In like manner we can derive $D^4(uv)$, etc.; and from the manner in which the coefficients arise it is evident that they will always be identical with those in the successive expansions of the powers of $a+b$; that is to say, they are the coefficients given in the Binomial Theorem. Hence

$$D^n(uv) = uD^nv + nDuD^{n-1}v + \frac{n(n-1)}{2}D^2uD^{n-2}v + \ldots$$

106. In particular, if we put $u = x$, $Du = 1$, the higher derivatives of u vanish, so that Leibnitz' Theorem reduces to its first two terms. Thus

$$D^n(xv) = \underbrace{Dx^nv}_{xD^nv} + nD^{n-1}v\,; \quad . \quad . \quad . \quad . \quad (1)$$

hence if we have the expression for the *n*th derivative of v, we can write that of xv.

For example, given $v = \log x$, using the expression for $D^r \log x$ found in Art. 102, we have

$$D^n(x \log x) = (-1)^{n-1}\left[\frac{(n-1)!}{x^{n-1}} - \frac{n(n-2)!}{x^{n-1}}\right],$$

which reduces to

$$D^n(x \log x) = (-1)^n \frac{(n-2)!}{x^{n-1}}.$$

This result is not applicable when $n = 1$, because the symbol $D_0 \log x$, which then occurs in the application of equation (1), cannot be evaluated by putting $r = 0$ in the expression for $D^r \log x$.

In like manner, if $u = x^2$, Leibnitz' Theorem reduces to its first three terms.

107. If $u = e^{ax}$, $D^n u = a^n e^{ax}$; that is to say, as applied to this simple function, the operation D has the same effect as multiplication by the constant a, and, since at each step the *operand*, or function operated on, remains of the same form, this is true of repeated operations. Now, using this value of u in Leibnitz' Theorem, we find

$$D^n(e^{ax}v) = e^{ax}\left[D^n + naD^{n-1} + \frac{n(n-1)}{2}a^2D^{n-2} + \dots\right]v, \quad (1)$$

in which the compound symbol prefixed to v means that the results of the operations of the several symbols upon v are to be added.

The result may be written in the form

$$D^n(e^{ax}v) = e^{ax}(D + a)^n v. \quad . \quad . \quad . \quad . \quad (2)$$

Here the symbol $D + a$ indicates the operation of taking the derivative and adding a times the operand itself, and the symbolic power indicates the repetition of this operation n times. In fact equation (2) may be derived directly from the value of the first derivative of $e^{ax}v$. For

$$\begin{aligned} D(e^{ax}v) &= e^{ax}Dv + ae^{ax}v \\ &= e^{ax}(D + a)v. \end{aligned}$$

The second member is of the same form as the original operand, $(D+a)v$ taking the place of the function v; hence, repeating the operation,

$$D^2(e^{ax}v) = e^{ax}(D + a)(D + a)v = e^{ax}(D + a)^2 v,$$

and so on for higher derivatives.

Examples XII.

Find the nth derivatives of the following functions:

1. $y = \dfrac{1}{(a - x)^m}$. $\qquad \dfrac{d^n y}{dx^n} = \dfrac{(m + n - 1)!}{(m - 1)!\,(a - x)^{m+n}}$.

2. $y = \log_b(a + x)$. $\qquad \dfrac{d^n y}{dx^n} = (-1)^{n-1} \dfrac{(n - 1)!}{\log b\,(a + x)^n}$.

3. $y = \log(1 - mx)$. $\qquad \dfrac{d^n y}{dx^n} = -(n - 1)!\,m^n(1 - mx)^{-n}$.

4. $y = e^{ax} \sin bx$. $\qquad \dfrac{d^n y}{dx^n} = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin\left[bx + n \tan^{-1}\dfrac{b}{a}\right]$.

5. $y = e^{x \cos \alpha} \cos(x \sin \alpha)$. $\qquad D^n y = e^{x \cos \alpha} \cos(x \sin \alpha + n\alpha)$.

6. $y = \cos^2 x$. $\qquad D^n y = 2^{n-1} \cos(2x + \tfrac{1}{2}n\pi)$.

7. $y = \dfrac{1}{a^2 - x^2} = \dfrac{1}{2a}\left[\dfrac{1}{a - x} + \dfrac{1}{a + x}\right]$.

$$D^n y = \frac{n!}{2a}\left[\frac{1}{(a - x)^{n+1}} + \frac{(-1)^n}{(a+x)^{n+1}}\right].$$

8. $y = \dfrac{x}{a^2 - x^2}$. $\qquad D^n y = \dfrac{n!}{2}\left[\dfrac{1}{(a-x)^{n+1}} - \dfrac{(-1)^n}{(a+x)^{n+1}}\right]$.

9. Prove that

$$D^{m+1}(x^m \log x) = \frac{m!}{x}.$$

10. $y = xe^{2x}$. $\qquad D^n y = 2^{n-1}(n + 2x)e^{2x}$.

11. $y = x^2 e^x$. $\qquad D^n y = [n(n-1) + 2nx + x^2]e^x$.

12. Prove that, when $n > 3$,

$$D^n(x^3 \log x) = (-1)^n \frac{6\,(n-4)!}{x^{n-3}}.$$

13. $y = x \sin x$. $\qquad \dfrac{d^n y}{dx^n} = x \sin\left(x + n\dfrac{\pi}{2}\right) - n \cos\left(x + n\dfrac{\pi}{2}\right)$.

14. If $y = \tan^{-1}x$, we have

$$(1 + x^2)\frac{dy}{dx} = 1;$$

hence derive the following relation between any three consecutive derivatives of $\tan^{-1}x$:

$$(1 + x^2)D^{n+1}\tan^{-1}x + 2nxD^n\tan^{-1}x + n(n-1)D^{n-1}\tan^{-1}x = 0.$$

15. If $y = \sin^{-1}x$, prove that

$$(1 - x^2)D^2y - xDy = 0;$$

and thence show that the higher derivatives satisfy the relation

$$(1 - x^2)D^{n+2}y - (2n+1)xD^{n+1}y - n^2D^ny = 0.$$

16. $y = (1 - x)^n x^n$.

$$\frac{d^n y}{dx^n} = n!\left\{(1-x)^n - n^2x(1-x)^{n-1} + \left[\frac{n(n-1)}{1.2}\right]^2 x^2(1-x)^{n-2} - \ldots\right\}.$$

17. If $y = x^m e^{ax}$, prove that

$$\frac{d^n y}{dx^n} = e^{ax}\left[a^n x^m + na^{n-1} m x^{m-1} + \frac{n(n-1)}{1 \cdot 2} a^{n-2} m(m-1) x^{m-2} + \dots\right],$$

and thence show that

$$x^{m-n} \frac{d^m}{dx^m}\left(e^{ax} x^n\right) = a^{m-n} \frac{d^n}{dx^n}\left(e^{ax} x^m\right).$$

18. From the nth derivative of $\tan^{-1}\frac{x}{a}$, Art. 104, derive that of $\frac{1}{a^2 + x^2}$.

$$D^n \frac{1}{a^2 + x^2} = (-1)^n \frac{n!}{a(a^2 + x^2)^{\frac{n+1}{2}}} \sin\left[(n+1) \cot^{-1}\frac{x}{a}\right].$$

19. If $y = \log \sqrt{(a^2 + x^2)}$, prove that

$$D^n y = (-1)^{n-1} \frac{(n-1)!}{(a^2 + x^2)^{\frac{n}{2}}} \cos\left(n \cot^{-1}\frac{x}{a}\right).$$

CHAPTER IV.

MAXIMA AND MINIMA.

XIII.

Characteristics of a Maximum Value.

108. ONE of the simplest applications of the Differential Calculus is the determination of the greatest and least values which a quantity varying continuously under given conditions can assume.

We suppose, at present, that the quantity can be expressed as a function of a single independent variable; so that the problem is that of determining the greatest or least value of a function $f(x)$ while x goes through a certain range of values.

For simplicity we shall always suppose x to *increase* through its range of values. Then, by Art. 39, $f(x)$ increases so long as the derivative $f'(x)$ is positive, and decreases so long as $f'(x)$ is negative. Hence, if z is a value of x for which $f(x)$ is a maximum, $f'(x)$ must change sign from $+$ to $-$, when x passes through the value z. Except in special cases, to be considered hereafter, $f'(x)$ is a continuous function, and therefore must take the value zero at the instant when it changes sign; hence z must be a value of x which satisfies the equation

$$f'(x) = 0.$$

109. For example, let it be required *to divide the number a into two such parts that the product of the square of one part and the cube of the other may have the greatest possible value.*

Taking the part to be squared for the independent variable x, the other part is $a - x$, and the quantity to be made a maximum is

$$f(x) = x^2(a - x)^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation $f'(x) = 0$ becomes in this case

$$2x(a - x)^3 - 3x^2(a - x)^2 = 0,$$

or

$$x(a - x)^2(2a - 5x) = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The roots of this equation are $x = 0$, $x = a$ and $x = \frac{2}{5}a$. The last value only corresponds to a division of a into two parts; it therefore gives the maximum required, and accordingly we find, on examining the first member of equation (2), that $f'(x)$ is positive when x is less than $\frac{2}{5}a$, and negative when x exceeds $\frac{2}{5}a$.

The maximum value of $f(x)$ is $f(\frac{2}{5}a)$, which, by substitution in equation (1), is $\frac{108}{3125}a^5$.

Maxima and Minima of Continuous Functions.

110. When a continuous function changes more than once from an increasing to a decreasing function, or vice versa, it is regarded as having a maximum or a minimum value whenever the change takes place. In other words, a value of a continuous function which is greater than the *neighboring* values is called a maximum, and one which is less than the neighboring values is called a minimum; even though greater values in the one case, or less values in the other, may exist.

For example, Fig. 21 is the graph of the function in equation (1) of the preceding article, so that the problem is to find the maximum or minimum

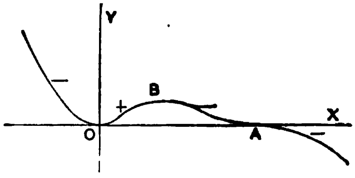

FIG. 21.

ordinates of the curve

$$y = x^2(a - x)^3,$$

which is continuous for all values of x. The point B in the diagram corresponds to the maximum value found above. The origin O and the point A, $(a,0)$ correspond to the other roots of equation (2). Now the first member of this equation

$$x(2a - 5x)(a - x)^2,$$

or value of $f'(x)$, is negative for negative values of x, and changes sign from $-$ to $+$ when x increases through zero. Accordingly, we have a minimum at the origin as well as a maximum at B; although there are values of y to the left of O greater than the maximum, and values to the right of A which are less than the minimum at O.

111. Since $f'(x) = \tan\phi$ the roots of the equation $f'(x) = 0$ are the abscissæ of the points on the curve where the gradient is zero, that is, where the tangent is parallel to the axis of x. They may be called *the critical values* of x because they are points at which maxima and minima may occur. But any one of them may fail to give either a maximum or a minimum. For example, in the present case, while x passes through a, $f'(x)$ becomes zero, but *does not change sign.* In fact $f'(x)$ after becoming negative at B remains negative, and the original function $f(x)$ continues to be a decreasing function for all values to the right of B.

The various sections of the curve are marked in the diagram with the sign of the derivative.

Maxima and Minima of Geometrical Magnitudes.

112. When the maximum or minimum value of a geometrical magnitude restricted by certain conditions is required, we seek if possible to obtain an expression for the magnitude in terms of a single unknown quantity, that is, to express it as a function of one independent variable.

For example: *let it be required to determine the cone of greatest convex surface among those which can be inscribed in a sphere whose radius is a.*

Any point A of the surface of the sphere being taken as the apex of the cone, let Fig. 22 represent a great circle of the sphere passing through the fixed point A.

If we refer the position of the point P in the base of the cone to rectangular coordinates, taking the centre of the sphere as origin, the required cone will evidently be determined when x is determined. We have now to express the convex surface S in terms of x.

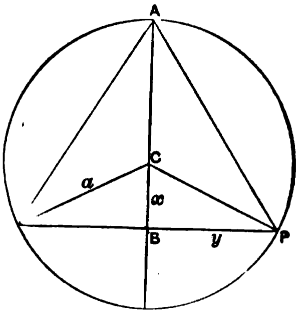

FIG. 22.

The expression for the convex surface of a cone gives

$$S = \pi y \sqrt{[y^2 + (a+x)^2]}, \quad . \quad . \quad . \quad . \quad (1)$$

in which the unknown quantities x and y are connected by the equation of the circle

$$x^2 + y^2 = a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting the value of y, we have

$$S = \pi \sqrt{(a^2 - x^2)} \sqrt{(2a^2 + 2ax)},$$

which reduces to

$$S = \pi \sqrt{(2a)}(a + x) \sqrt{(a - x)}. \quad . \quad . \quad . \quad . \quad (3)$$

Since the factor $\pi\sqrt{(2a)}$ is constant, we are evidently required to find the value of x for which the function

$$f(x) = (a + x)\sqrt{(a - x)}$$

is a maximum. The equation $f'(x) = 0$ is, in this case,

$$\sqrt{(a - x)} - \frac{a + x}{2\sqrt{(a - x)}} = 0;$$

whence

$$x = \tfrac{1}{3}a.$$

The altitude of the required cone is therefore $\frac{4}{3}a$. Substituting the value of x in equation (3), we have

$$S = \tfrac{8}{9}\sqrt{3}\cdot\pi a^2,$$

the maximum value required.

113. As a further illustration, let it be required *to determine the greatest cylinder that can be inscribed in a given segment of a paraboloid of revolution.*

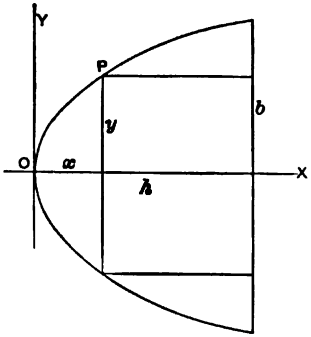

FIG. 23.

Let h denote the altitude, and b the radius of the base of the segment. The equation of the generating parabola is of the form

$$y^2 = 4ax.$$

Since (h,b) is a point of the curve, we have the condition $b^2 = 4ah$; hence, eliminating $4a$, the equation of the curve is

$$y^2 = \frac{b^2}{h}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The volume V of the cylinder of which the maximum value is required is expressed by $V = \pi y^2(h - x)$, or, by equation (1),

$$V = \pi \frac{b^2}{h} x(h - x).$$

Hence we put

$$f(x) = hx - x^2,$$

and the condition $f'(x) = 0$ gives

$$x = \tfrac{1}{2}h.$$

Consequently $h - x$, the altitude of the cylinder, is one-half the altitude of the paraboloid.

114. A problem involving a maximum or minimum sometimes requires statement in a changed form, before the variable can be made a function of a single independent variable. For example: required the length of the longest rod which can be passed up a chimney of which the width is b and the height of the opening above the floor is a, the rod being supposed to lie in a vertical plane, see Fig. 24. Taking as coordinate axes the intersections, OA and OB, of this plane with the floor and the vertical back wall of the chimney, it is obvious that the rod cannot be greater than any line AB passing through the point (a, b) and terminated by the axes. The length required is therefore the same as the *minimum* length of the line AB. This length may now be expressed in terms of θ, its inclination to the floor. Thus

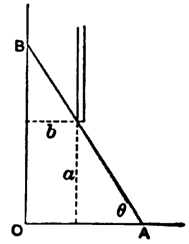

FIG. 24.

$$AB = f(\theta) = a \operatorname{cosec} \theta + b \sec \theta.$$

Hence, putting

$$f'(0) = -a \operatorname{cosec}\theta \cot\theta + b \sec\theta \tan\theta = 0,$$

we obtain

$$\tan^3\theta = \frac{a}{b},$$

and substituting, we have for the minimum value of AB, or maximum length of the rod,

$$(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}.$$

Examples XIII.

1. Find the sides of the largest rectangle that can be inscribed in a semicircle of radius a. The sides are $a\sqrt{2}$ and $\frac{1}{2}a\sqrt{2}$.

2. Determine the maximum right cone inscribed in a given sphere. The altitude is four-thirds of the radius of the sphere.

3. Determine the maximum rectangle inscribed in a given segment of a parabola.

The altitude of the rectangle is two-thirds that of the segment.

4. Find the maximum cone of given slant height a.

The radius of the base is $\frac{1}{3}a\sqrt{6}$.

5. A boatman 3 miles out at sea wishes to reach in the shortest time possible a point on the beach 5 miles from the nearest point of the shore; he can pull at the rate of 4 miles an hour, but can walk at the rate of 5 miles an hour; find the point at which he should land.

Express the whole time in terms of the distance of the required point from the nearest point of the shore.

He should land one mile from the point to be reached.

6. If a square piece of sheet lead whose side is a have a square cut out at each corner, find the side of the latter square in order that the remainder may form a vessel of maximum capacity.

The side of the square is $\frac{1}{6}a$.

7. A rectangular court is to be built so as to contain a given area c^2, and a wall already constructed is available for one of the sides; find its dimensions so that the least expense may be incurred.

The side parallel to the wall is double each of the others.

8. Determine the maximum cylinder inscribed in a given cone.

The altitude of the cylinder is one-third that of the cone.

9. Find the maximum cylinder that can be inscribed in a sphere whose radius is a. The altitude is $\frac{2}{3}a\sqrt{3}$.

10. Through a point whose rectangular coordinates are a and b draw a line such that the triangle formed by this line and the coordinate axes shall have a minimum area.

The intercepts on the axes are $2a$ and $2b$.

11. The illumination of a plane surface by a luminous point varies inversely as the square of its distance from the point, and directly as the cosine of the angle of incidence of the rays; find the height at which a bracket-burner must be placed, in order that a point on the floor of a room at the horizontal distance a from the burner may receive the greatest possible amount of illumination.

The height is $\frac{a}{\sqrt{2}}$.

12. A cylinder is inscribed in a cone whose altitude is a, and the radius of whose base is b; determine the cylinder so that its total surface shall be a maximum, and thence show that there will be no maximum when $a < 2b$.

The altitude is $\frac{a^2 - 2ab}{2(a-b)}$.

13. Determine the cone of minimum volume described about a given sphere. The height is twice the diameter of the sphere.

14. A sphere has its centre in the surface of a given sphere whose radius is a; determine its radius in order that the area of the surface intercepted by the given sphere may be a maximum. $\frac{4}{3}a$.

15. Find the point, on the line joining the centres of two spheres whose radii are a and b, from which the greatest amount of spherical surface is visible.

The distance between the centres is divided in the ratio $a^{\frac{3}{2}} : b^{\frac{3}{2}}$.

16. Find the minimum isosceles triangle circumscribed about a parabolic segment.

The altitude of the triangle is four-thirds of the altitude of the segment.

17. A tinsmith was ordered to make an open cylindrical vessel of given volume, which should be as light as possible; find the ratio between the height and the radius of the base.

The height should equal the radius of the base.

18. What should be the ratio between the diameter of the base and the height of cylindrical fruit-cans in order that the amount of tin used in constructing them may be the least possible?

The height should equal the diameter of the base.

19. Assuming that the expenditure of coal in driving a steamer through the water is proportional to the time and to the cube of the speed v, find the most economical speed against a current whose speed is a. $v = \frac{3}{2}a$.

20. In Fig. 24, find the minimum value of the sum of the intercepts OA and OB. $(\sqrt{a} + \sqrt{b})^2$.

21. Find the minimum perimeter of the triangle OAB in Fig. 24.

$2[a + b + \sqrt{(2ab)}]$.

22. A right cone is cut by a plane parallel to the slant height AB. Given that the section is a parabola, and that the area of a parabola is $\frac{2}{3}$ of the circumscribing rectangle; prove that the area is a maximum when the plane bisects the radius OB.

23. From a point whose abscissa is c, on the axis of the parabola $y^2 = 4ax$, determine the shortest line to the curve.

The abscissa of the required point on the curve is $c - 2a$.

24. Determine the greatest rectangle that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The sides are $a\sqrt{2}$ and $b\sqrt{2}$.

25. The top of a pedestal which sustains a statue a feet in height is b feet above the level of a man's eyes; find his horizontal

distance from the pedestal when the statue subtends the greatest angle.

$$\sqrt{[b(a+b)]}.$$

26. It is required to construct from two circular iron plates of radius a a buoy, composed of two equal cones having a common base, which shall have the greatest possible volume.

The radius of the base $= \frac{1}{3}a\sqrt{6}$.

27. In a given sphere, determine the inscribed cylinder whose entire surface is a maximum.

Solution :—

Using the notation of Art. 112, we find

$$f(x) = a^2 - x^2 + 2x\sqrt{(a^2 - x^2)};$$

whence $$f'(x) = -2x + 2\sqrt{(a^2 - x^2)} - \frac{2x^2}{\sqrt{(a^2 - x^2)}},$$

and $f'(x) = 0$ gives

$$x\sqrt{(a^2 - x^2)} = a^2 - 2x^2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Squaring, we have

$$5x^4 - 5a^2x^2 + a^4 = 0,$$

the roots of which are

$$x^2 = a^2(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\tfrac{1}{5}});$$

but, since the radical in equation (1) must be positive, we must have $x^2 < \frac{1}{2}a^2$; hence the altitude, $2x$, of the cylinder is

$$a\sqrt{(2 - \tfrac{2}{5}\sqrt{5})}.$$

28. In a given sphere determine the inscribed cone whose entire surface is a maximum.

The altitude of the cone is $\frac{a}{16}(23 - \sqrt{17})$.

XIV.

Discrimination between Maxima and Minima.

115. We have seen in Arts. 110 and 111 that, in the case of a continuous function, the equation $f'(x) = 0$ may have a number of roots, which are the critical values of x to be examined for the occurrence of maxima and minima; and that, in the graph of the function, these correspond to points where the curve is parallel to the axis of x. If one of these occurs in a part of the curve which is *convex* as viewed from above, as for example B in Fig. 21, the ordinate $f(x)$ is there a *maximum*. If it occurs, like O in Fig. 21, in a concave part of the curve, $f(x)$ is a minimum. Finally, if it occurs at a *point of inflection*, like A in Fig. 21, there is at the point neither a maximum nor a minimum.

116. It was shown in Art. 98 that, when the value of the second derivative $f''(x)$ is *negative*, the curve $y = f(x)$ is *convex*, and when it is positive, the curve is concave. Accordingly, if $f''(x)$ has a *negative* value for a critical value of x, we have a *maximum* value of $f(x)$; and if $f''(x)$ has a *positive* value, $f(x)$ is a *minimum*. Thus if $f''(x)$ has a finite value at a critical point (that is, a point at which $f'(x) = 0$), a maximum or minimum occurs; but, if $f''(x) = 0$, it is necessary to make a further examination to ascertain whether there is or is not a point of inflection.

117. For this purpose, we notice that at any point of inflexion the gradient $f'(x)$ is either a maximum or a minimum. For example, in Fig. 20 $f'(x)$ has a minimum value at the point of inflexion, while in Fig. 21 its value at A is a maximum. It follows that, for a point of inflexion, the derivative of $f'(x)$, which is $f''(x)$, must not only vanish but *must change* *sign*. Therefore, by Art. 116, if $f'''(x)$ has a finite value, *there will be a point of inflexion;* and in that case the original function, $f(x)$, will have *neither a maximum nor a minimum value* at the critical point in question.

118. In the next place, if $f'''(x)$ vanishes at the critical point as well as $f'(x)$ and $f''(x)$, we examine $f^{\text{iv}}(x)$. If this has a finite value, $f'(x)$, of which it is the third derivative, will, as shown above, *not* have a maximum or minimum value; hence there will be no point of inflexion at the critical point, and the original function $f(x)$ *will* have a maximum or minimum value.

Continuing in this way, we can prove that, whenever the first one of the successive derivatives which does not vanish is of an *even order*, *there will be a maximnm or minimum value;* but, if it is of an *odd order*, there will be a point of inflexion, and hence *no maximum or minimum value.*

In other words, if all the derivatives of $f(x)$ preceding $f^n(x)$ vanish for a certain value of x, while $f^n(x)$ has a finite value, $f(x)$ will have a maximum or minimum value if n is even, but not if n is odd.

119. In the next place, supposing n to be even, we shall show that to discriminate between a maximum and a minimum we have the same rule, depending on the sign of $f^n(x)$, as in the case when $n = 2$, Art. 116.

To prove this, we notice: first, that at a horizontal point of inflexion where $f'(x)$ is a *maximum* (like A in Fig. 21), the function $f(x)$ is a *decreasing* one. Secondly, when at a horizontal point $f'(x)$ is a *decreasing* function, so that it changes its value in the order $+$, 0, $-$, $f(x)$ is a *maximum.* It follows that when $f^n(x)$ is *negative,* the preceding functions are alternately decreasing ones and maxima. In like manner, when $f^n(x)$ is *positive,* the functions are alternately *increasing* functions and *minima*. Thus when n is even, a *positive* value indicates a *minimum* and a *negative* value a *maximum*.

120. As an illustration, let us take the function

$$f(x) = e^x + e^{-x} + 2\cos x,$$

whence

$$f'(x) = e^x - e^{-x} - 2\sin x.$$

In this case, $f'(x) = 0$ is a transcendental equation, but it is obvious that $x = 0$ is a root. We therefore examine the values of $f''(0)$, etc. Differentiating again,

$$f''(x) = e^x + e^{-x} - 2\cos x, \quad \therefore \quad f''(0) = 0;$$

$$f'''(x) = e^x - e^{-x} + 2\sin x, \quad \therefore \quad f'''(0) = 0;$$

$$f^{\text{iv}}(x) = e^x + e^{-x} + 2\cos x, \quad \therefore \quad f^{\text{iv}}(0) = 4.$$

The fourth derivative is the first one which does not vanish, and it has a positive value; we therefore conclude that $f(0)$ is a minimum value of $f(x)$.

Alternation of Maxima and Minima.

121. It is obvious that, in the case of a continuous function, maxima and minima (when several exist) must occur alternately. This fact facilitates the discrimination of these values. For example, given

$$f(x) = 3x^4 - 16x^3 - 6x^2 + 12,$$

which is continuous for all values of x. Here

$$f'(x) = 12x^3 - 48x^2 - 12x.$$

The roots of $f'(x) = 0$ are $x = 0$ and $x = 2 \pm \sqrt{5}$. Again,

$$f''(x) = 36x^2 - 96x - 12.$$

For the root $x = 0$, we find $f''(0) = -12$; therefore $x = 0$ gives a ***maximum.***

Now, of the other two roots one is positive and the other negative, so that zero is the intermediate root. It follows that each of these roots gives a minimum of the function.

122. The same conclusion may be arrived at, in this case, as follows: Very large positive values of x make $f(x)$ very large and positive; hence for the range of values of x beyond the greatest critical value, which is $x = 2 + \sqrt{5}$, $f(x)$ is an increasing function. Therefore this value of x corresponds to a ***minimum.*** These conclusions are made clear by means of a rough sketch of the graph of the function.*

123. It is obvious also that, in the case of a continuous function, a maximum must be greater than an adjacent minimum and a minimum less than an adjacent maximum. But neither this conclusion nor the alternation of maxima and minima can be inferred of the maxima and minima occurring in different branches of a discontinuous function.

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* In the same way, we can see that the function in Art. 120 must have at least one minimum; for its values increase indefinitely and are positive both for large positive and for large negative values of x. Hence, if zero is the only critical value, it must correspond to a minimum. Moreover, that there is no other critical value except zero may be shown as follows: $f''(x)$ may be put in the form

$$f''(x) = \frac{e^{2x} - 2\cos x \,.\, e^x + 1}{e^x} > \frac{e^{2x} - 2e^x + 1}{e^x};$$

but the last expression (of which the numerator is a perfect square) cannot become negative. Therefore $f'(x)$ cannot again become zero.

Employment of a Substituted Function.

124. It is often convenient, in determining a maximum or minimum, to substitute for the given variable some function of it which obviously arrives at its maximum or minimum at the same time. For example, to determine the maximum value of

$$y = \sqrt{(b^2 + ax)} + \sqrt{(b^2 - ax)}.$$

It is obvious that the square of a positive quantity will reach a maximum simultaneously with the quantity itself. In this case

$$y^2 = 2b^2 + 2\sqrt{(b^4 - a^2x^2)},$$

which is obviously a maximum when $x = 0$. We infer that y is a maximum when $x = 0$; the maximum value is therefore $2b$.

125. A decreasing function of a variable (that is, one which decreases when the variable increases and increases when the variable decreases) will evidently reach a maximum when the original variable reaches a minimum, and *vice versa.* Thus, to find the maxima and minima of

$$f(x) = \frac{x}{x^2 - 3x + 1};$$

we may with advantage employ the reciprocal, viz.,

$$\phi(x) = x - 3 + \frac{1}{x}.$$

Taking derivatives,

$$\phi'(x) = 1 - \frac{1}{x^2}, \qquad \phi''(x) = 2x^{-3}.$$

The roots of $\phi'(x) = 0$ are $x = \pm 1$; $x = 1$ makes $\phi''(x)$

positive; hence it gives a minimum value of $\phi(x)$, and therefore a ***maximum*** value of $f(x)$. In like manner, $x = -1$ is found to give a maximum value of $\phi(x)$, and therefore a ***minimum*** of $f(x)$.

In this example, the maximum value, which is $f(1) = -1$, is algebraically less than the minimum, which is $f(-1) = -\frac{1}{5}$. This is accounted for by the fact that the function is discontinuous; it has an infinite value corresponding to

$$x = \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5} = .38,$$

a value which lies between $+1$ and -1; so that the maximum and minimum points occur in different branches of the graph of the function.

126. In some cases we may use a simplified function in place of $f'(x)$ in discriminating between maxima and minima. For example, given the function

$$f(x) = \frac{x}{\log x}, \qquad \text{whence} \qquad f'(x) = \frac{\log x - 1}{(\log x)^2}.$$

Since our object is only to ascertain whether $f'(x)$ changes sign in the order $-$, 0, $+$ or in the order $+$, 0, $-$, we may omit the denominator, which is always positive. The numerator, $\log x - 1$, vanishes when $x = e$, and its derivative which takes the place of $f''(x)$, namely x^{-1}, is ***positive*** when $x = e$; hence the corresponding value of the function, namely $f(e) = e$, is a ***minimum***.

Examples XIV.

1. Show that the function $ae^{kx} + be^{-kx}$ has a minimum value equal to $2\sqrt{(ab)}$.

Determine the maxima and minima of the following functions:

2. $f(x) = x^x$. A min. for $x = \frac{1}{e}$.

3. $f(x) = \dfrac{\log x}{x^n}$. A max. for $x = \dfrac{1}{e^n}$.

4. $f(x) = \dfrac{(a-x)^3}{a-2x}$. A min. for $x = \frac{1}{4}a$.

5. $f(x) = \dfrac{1+3x}{\sqrt{(1+5x)}}$. A min. for $x = -\frac{1}{15}$.

6. $f(x) = 2\cos x + \sin^2 x$. Max. for $x = 2n\pi$;
min. for $x = (2n+1)\pi$.

7. $f(x) = \sin x(1+\cos x)$. Max. for $x = \frac{1}{3}\pi$;
min. for $x = -\frac{1}{3}\pi$;
neither for $x = \pi$..

8. $f(x) = \sec x + \log \cos^2 x$. Max. for $x = 0$, and $x = \pi$;
min. for $x = \pm \frac{1}{3}\pi$.

9. $f(x) = \dfrac{\tan^3 x}{\tan 3x}$. Min. for $x = 0$, $\frac{3}{8}\pi$, $\frac{5}{8}\pi$, and π;
max. for $x = \frac{1}{8}\pi$, $\frac{1}{2}\pi$, $\frac{7}{8}\pi$, etc.

10. $f(x) = e^x + e^{-x} - x^2$. A min. for $x = 0$.

11. $f(x) = 4x^2 + \cos 2x - \frac{1}{2}(e^{2x} + e^{-2x})$.
Max. for $x = 0$.

12. $f(x) = (3-x)e^{2x} - 4xe^x - x$. Is there a maximum or a minimum corresponding to $x = 0$? Neither.

13. $f(x) = x(x+a)^2(x-a)^3$.
Min. for $x = -a$ and $x = \frac{1}{3}a$;
max. for $x = -\frac{1}{2}a$.

14. $f(x) = \sin 2x - x$. Max. for $x = n\pi + \frac{1}{6}\pi$;
min. for $x = n\pi - \frac{1}{6}\pi$.

15. $f(x) = 2x^3 + 3x^2 - 36x + 12$. Max. for $x = -3$;
min. for $x = 2$.

16. $f(x) = x^3 - 3x^2 - 9x + 5$. Max. for $x = -1$;
min. for $x = 3$.

17. $f(x) = 3x^5 - 125x^3 + 2160x$.
Max. for $x = -4$ and $x = 3$;
min. for $x = -3$ and $x = 4$.

18. $f(x) = b + c(x - a)^{\frac{5}{3}}$. Neither a max. nor a min.

19. $f(x) = (x - 1)^4(x + 2)^3$. Max. for $x = -\frac{5}{7}$; min. for $x = 1$.

20. $f(x) = (x - 9)^5(x - 8)^4$. Max. for $x = 8$; min. for $x = 8\frac{4}{9}$.

21. $f(x) = \dfrac{1 - x + x^2}{1 + x - x^2}$. Min. for $x = \frac{1}{2}$.

22. $f(x) = \dfrac{ax}{ax^2 - bx + a}$. Max. for $x = 1$; min. for $x = -1$.

23. $f(x) = \dfrac{x - 1}{x^3 - 3x^2 + 2x + 54}$. Max. for $x = 4$.

24. $f(x) = \dfrac{x^2 - x + 1}{x^2 + x - 1}$. Max. for $x = 0$; min. for $x = 2$.

25. The lower corner of a leaf of a book is folded over so as just to reach the inner edge of the page. Denoting by a the width of the page, and by x the part of the lower edge turned over, show that the length of the crease is

$$y = \frac{x\sqrt{x}}{\sqrt{(x - \frac{1}{2}a)}};$$

and thence find x when y is a minimum. $x = \frac{3}{4}a$.

26. Find when the area of the part folded over is a minimum. $x = \frac{2}{3}a$.

XV.

Implicit Functions.

127. When y is an implicit function of x, defined by the equation

$$F(x, y) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the first derivative, found as in Art. 87, takes the form

$$\frac{dy}{dx} = \frac{u}{v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where u and v are usually functions of x and y. This derivative takes the value zero if $u = 0$, provided v does not vanish at the same time.* Hence, to find a maximum or minimum value of y, we must find the values of x and y which satisfy simultaneously the two equations

$$F(x, y) = 0 \qquad \text{and} \qquad u = 0.$$

This is the same thing as finding the *horizontal* points of the curve whose rectangular equation is $F(x, y) = 0$.

128. For example, let us take the equation

$$xy^2 - x^2y = 2a^3, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

in which a denotes a positive constant. Differentiating,

$$y^2 + 2xy\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0;$$

therefore,

$$\frac{dy}{dx} = \frac{y(2x - y)}{x(2y - x)}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

In this example, $u = y(2x - y)$ and $v = x(2y - x)$; putting $u = 0$, we obtain

$$y = 0 \qquad \text{or} \qquad y = 2x.$$

Substituting $y = 0$ in equation (1) gives an infinite value of x, showing that the curve has the axis of x for an asymptote as represented in Fig. 25.

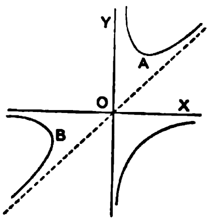

FIG. 25.

Next combining $y = 2x$ with equation (1), we find

$$x = a \qquad \text{and} \qquad y = 2a,$$

which are the coordinates of the point A in the diagram. At this point v does not vanish, therefore the curve has a horizontal tangent, the ordinate in this case being a minimum.

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* The case in which u and v vanish simultaneously will be considered in the next chapter. See Art. 171

129. When it is necessary to find the value of the second derivative at a horizontal point in order to discriminate between maxima and minima, the work of finding it, as illustrated in Art. 100 for the general case, can be much shortened. Differentiating equation (2), Art. 127, with respect to x, we have

$$\frac{dy^2}{dx^2} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2};$$

but, since, in the cases now under consideration, $u = 0$, the second term in the numerator vanishes. Hence, distinguishing by brackets the special values which the derivatives take when $\dfrac{dy}{dx} = 0$, we have

$$\left[\frac{d^2y}{dx^2}\right] = \frac{\left[\dfrac{du}{dx}\right]}{v},$$

in which the values of x and y found for the horizontal point are to be substituted. For example, in the illustration given in Art. 128, we find

$$\left[\frac{d^2y}{dx^2}\right] = \frac{2y}{x(2y - x)}; \quad \text{whence} \quad \left.\frac{d^2y}{dx^2}\right]_{a,\,2a} = \frac{4}{3a},$$

which, having a *positive* value, indicates a *minimum* ordinate as in the diagram.

Maximum and Minimum Abscissæ.

130. If, in equation (1), Art. 127, we regard x as an implicit function of y, we have, using the same notation,

$$\frac{dx}{dy} = \frac{v}{u}.$$

Hence the points at which x has a maximum or minimum value are found by means of the simultaneous equations

$$F(x, y) = 0 \quad \text{and} \quad v = 0.$$

For instance, in the example of Art. 128, $v = 0$ gives $x = 0$ or $x = 2y$. Combining these in turn with equation (1), the first gives the infinite value of y, indicating the axis of y as an asymptote; the second gives $x = -2a$, $y = -a$, the coordinates of the point B in Fig. 25, at which the abscissa is a maximum. Points of this character, where the tangent to the curve is parallel to the axis of y, may be called the *vertical* points of the curve.

Infinite Values of the Derivative.

131. When x is regarded as the independent variable and y as the function, the *vertical* points are those at which the derivative $\frac{dy}{dx}$ takes an infinite value. They are usually points like B in Fig. 25, at which the function y is *discontinuous*. Thus, in the figure, y is a two-valued function for values of x less than $-2a$. The two values become equal when $x = -2a$, and become imaginary for values greater than $-2a$.

In fact, whenever x regarded as a function of y has a maximum or minimum value, it is evident that the curve lies on one side of the tangent in the neighborhood of the point of contact; hence this value of x is, for the inverse function y, *the limit of a range of values for which that function is continuous.*

When the equation is quadratic for y this gives a convenient method of finding maxima or minima values of x.

132. There are, however, two exceptional cases in which a vertical point does not give a limiting value of x, the function y remaining continuous when x *passes through* the value in question. In other words, there are cases in which the curve crosses the vertical tangent.

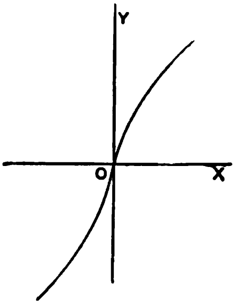

Fig. 26.

The first case is that in which the point of contact is also a point of inflexion. For example, in the case of the function

$$y = \sqrt[3]{x} \qquad \text{or} \qquad y^3 = x,$$

we have $\dfrac{dy}{dx} = \dfrac{1}{3x^{\frac{2}{3}}}$, which is infinite at the origin, but is real and positive on both sides of the origin. The curve takes the form given in Fig. 26, neither x nor y having a maximum or minimum value.

133. The second case is that in which the curve lies on both sides of the tangent at the critical point, but upon the same side of the normal. In this case, the curve is said to have a *cusp*. For example, in the case of the function

$$y = x^{\frac{2}{3}} \qquad \text{or} \qquad y^3 = x^2,$$

we have $\dfrac{dy}{dx} = \dfrac{2}{3x^{\frac{1}{3}}}$, which, as in the preceding case, is infinite at the origin and is real on both sides of the origin, that is for positive and negative values of x. There is, in this case, a *minimum value of y* because the derivative $\frac{dy}{dx}$ changes sign from $-$ to $+$ as x passes through the value 0. The curve, which is *the semi-cubical parabola*, takes the form given in Fig. 27.

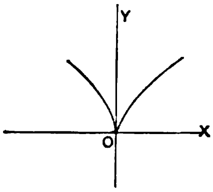

FIG. 27.

This is the exceptional case mentioned in Art. 108 in which a maximum or minimum occurs although the derivative is not zero. The general condition for a case of this kind is that when the derivative is infinite, the function y shall remain finite and continuous, and that the derivative shall *change sign*. The function y is then a maximum or minimum according as the change is from $+$ to $-$ or from $-$ to $+$, exactly as in the usual case.*

Functions of Two Variables.

134. A maximum value of a function $f(x, y)$ of two independent variables is defined as a value greater than any neighboring value of the function. In other words, $f(a, b)$ is a maximum value, if $f(x, y)$ changes from a state of increasing to a state of decreasing when x and y pass simultaneously through the values $x=a$ and $y=b$, irrespective of the relative value of their rates. In particular, if either of the variables is assumed constant, the conditions for a maximum must be fulfilled when the other varies.

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* On the other hand, if in such a case we were considering x as a function of y, we should find the derivative $\frac{dx}{dy} = 0$; but neither x nor its derivative would be a continuous function while y passes through the critical value. Thus, in the example, $x = y^{\frac{3}{2}}$ and $\frac{dx}{dy} = \frac{3}{2}\sqrt{y}$, both of which become imaginary when y passes through the critical value zero.

Similar remarks of course apply to a minimum value.

135. When a maximum obviously exists, it is easy in this way to obtain two relations which must exist between the variables, and thus determine the special values of x and y. For example, let it be required to divide a number a into three parts such that their product shall be a maximum. Here two of the parts are independent variables; but assigning to one of them *any* fixed value, it is easily shown that the other two parts must be equal if the product is a maximum. It follows that all three parts must be equal and therefore each part is $\frac{1}{3}a$.

Again, let it be required to inscribe the maximum parallelopiped in a given cone. Here, supposing the height to have any fixed value, it is obvious that the base must be the maximum rectangle inscribed in a given circle, which is a square. Now when the altitude varies, the parallelopiped with a square base bears a fixed ratio to the circumscribing cylinder which is itself inscribed in the cone. Hence the altitude is the same as that of the maximum cylinder inscribed in a cone, which is readily found to be one-third the altitude of the cone.

Critical Points on a Surface.

136. In general, if we put

$$z = f(x, y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in order that $f(a, b)$ shall be a maximum or minimum value of z, it is *a necessary but not a sufficient condition* that the derivative $\frac{dz}{dx}$ shall change sign when $y = b$ and x passes through the value a, and also that $\frac{dz}{dy}$ shall change sign when $x = a$ and y passes through the value b. We shall consider only the usual case in which each of these derivatives takes the value zero. Thus the special values of x and y will be found among those which simultaneously satisfy the two equations

$$\frac{dz}{dx} = 0 \qquad \text{and} \qquad \frac{dz}{dy} = 0. \quad . \quad . \quad . \quad (2)$$

137. If we regard x, y and z as the three rectangular coordinates of a point, and consider the surface represented by equation (1), the problem before us becomes that of finding the points on the surface which are at a maximum or minimum distance from the plane of xy. The points on the surface of which the x and y coordinates satisfy equations (2) are those at which the tangent plane is parallel to the plane of xy. These are the *critical points* at which maxima or minima *may* occur. But for a *maximum* it is further necessary that the surface shall lie below the tangent plane at least for a certain region, in the neighborhood of, and completely surrounding, the critical point. That is, *the surface must be convex as viewed from above.*

138. When this is the case, the sections of the surface made by planes parallel to that of xz, and *passing through or near to* the critical point, will also be convex as viewed from above. The equation of a section of this kind is simply the equation of the surface when y is regarded as a constant and z as a function of x only. Hence the convexity of these curves is equivalent to the condition that z shall fulfil the requirements of a maximum when regarded as a function of x.

It is of course also necessary that z should fulfil the conditions of a maximum when regarded as a function of y.

In like manner a minimum function of two variables must fulfil the conditions for a minimum, both when regarded as a function of x and when regarded as a function of y.

139. When examining a point in a given example, it must be remembered however that, although the above conditions are necessary, they are not of themselves sufficient to establish the existence of a maximum or minimum.

But if we determine the value of x for which z is a maximum in terms of y, and substitute this in place of x in the expression for z, we shall have a function of y which represents the greatest of the values of z in each of the several sections of the surface made by planes parallel to the plane of xz; and if it can *then* be shown that this function assumes its maximum value when $y = b$, it will have been completely demonstrated that this value is a true maximum. So also, *mutatis mutandis*, in the case of a minimum, as illustrated in the following example:

140. Given the function

$$z = x^3 - 3axy + y^3, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence

$$\frac{dz}{dx} = 3x^2 - 3ay, \qquad \frac{dz}{dy} = 3y^2 - 3ax.$$

From the simultaneous equations

$$3x^2 - 3ay = 0 \qquad \text{and} \qquad 3y^2 - 3ax = 0 \quad . \quad . \quad (2)$$

we find the critical values (a, a) and $(0, 0)$. In order to test the values (a, a), we form the second derivative of z as a function of x, namely

$$\frac{d^2z}{dx^2} = 6x.$$

This has a positive value for every point near the critical point (a, a), indicating a minimum. Now, from the first of equations (2), the value of x which makes z a minimum is $x = \sqrt{(ay)}$. Substituting this in equation (1), we have for the minima values of z corresponding to different values of y,

$$z = y^3 - 2a^{\frac{3}{2}}y^{\frac{3}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

whence

$$\frac{dz}{dy} = 3y^2 - 3a^{\frac{3}{2}}y^{\frac{1}{2}}, \qquad \frac{d^2z}{dy^2} = 6y - \tfrac{3}{2}a^{\frac{3}{2}}y^{-\frac{1}{2}}.$$

The first derivative is zero when $y = a$ as before found and the second derivative has a positive value. This indicates that the value of z corresponding to $y = a$ is a minimum among the minima represented by equation (3), therefore it is a true minimum.

Considering next the critical values (0, 0), we find that the second derivative with respect to x vanishes for $x = 0$, and since the third derivative has the finite value 6, there is, according to Art. 117, neither a maximum nor a minimum at the origin.

Examples XV.

1. Given $25y^2 - 6xy + x^2 - 9 = 0$, determine the maxima and minima of y. Min. for $x = -\frac{9}{4}$; max. for $x = \frac{9}{4}$.

2. Given $x^4 + 2ax^2y - ay^3 = 0$, find the maxima and minima of y. Min. for $x = \pm a$.

3. Given $y^2 - x^2y + x - x^3 = 0$, prove that $x = -1$ gives a maximum value of y.

4. Given $3a^2y^2 + xy^3 + 4ax^3 = 0$. Show that when $x = \frac{3}{2}a$, y has a maximum value, namely $-3a$.

5. Given $y^3 + x^3 - 3axy = 0$, find maxima and minima of y.
Max. for $x = a\sqrt[3]{2}$;
min. for $x = 0$.

6. Given $x^2(y^2+1) = 4H(xy+h)$, find the maximum value of x by the method mentioned in Art. 131. $x = 2\sqrt{(H^2+Hh)}$.

Find maxima and minima of the following functions:

7. $f(x) = (x^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{1}{3}}$. Min. for $x = 0$.

8. $f(x) = (x^2 - b^2)^{\frac{2}{3}}$. Max. for $x = 0$;
min. for $x = \pm b$.

9. $f(x) = (x^2 + 3x + 2)^{\frac{2}{3}} + x^{\frac{2}{3}}$.
$f'(x) = \infty$ gives min. corresponding to $x = -2$, $x = -1$ and $x = 0$.
$f'(x) = 0$ gives two intermediate maxima.

10. $f(x) = (x^2 + 2x)^{\frac{2}{3}} - (x+3)^{\frac{4}{3}}$. Max. for $x = \frac{1}{4}(-3 \pm \sqrt{17})$;
min. for $x = 0$ and $x = -2$.

11. $f(x) = (x-a)^{\frac{4}{3}}(x-b)^{\frac{2}{3}} + c$. Max. for $x = \frac{2b+a}{3}$;
min. for $x = a$ and $x = b$.

12. $f(x) = \frac{(x-a)(x-b)}{x^2}$. Min. for $x = \frac{2ab}{a+b}$.

13. $f(x) = (x-a)^{\frac{2}{3}}(x-b)^{\frac{1}{3}}$.

Solutions for $x = a$ and $x = \frac{1}{3}(2b + a)$; if $b > a$, the former gives a max. and the latter a min.

14. A number is to be divided into three parts, such that the product of the *m*th power of the first, the *n*th power of the second, and the *p*th power of the third shall be a maximum. Show that the parts will be in the ratios $m : n : p$.

15. Show that the polygon of given perimeter and number of sides has a maximum area when equilateral and equiangular.

16. Show that the function

$$x^2 - 3xy + 2y^2 + 4x - 2y + 3$$

has a minimum value corresponding to $x = 10, y = 8$.

17. Show that the function

$$a^2 + b^2 - x^2 - y^2 - 2b\sqrt{(a^2 - y^2)}$$

has a maximum when $a > b$, but neither max. nor min. when $a < b$ and when $a = b$.

18. Determine whether

$$z = \sqrt{a^2 - y^2 + x^2}$$

has a maximum or minimum value.

CHAPTER V.

EVALUATION OF INDETERMINATE FORMS.

XVI.

Indeterminate or Illusory Forms.

141. A QUANTITY given in the form of a fraction is ***indeterminate in value*** if both numerator and denominator have the value zero and admit of no other value.

Suppose now that the terms of the fraction are continuous functions of x which becomo zero for a particular value, say a, of x (but are in general not equal to zero); then the fraction is itself a continuous function of x, and is said to take *the indeterminate form* $\frac{0}{0}$, when $x = a$. Such a fraction has definite values, which for all values of x except a can be found by division. As x approaches indefinitely near to a, these values approach indefinitely near to a certain value which is often called *the limiting value* of the fraction; and, in order to make the fraction a *continuous function of x when x passes through a,* it is necessary to regard this limiting value as the value of the fraction corresponding to $x = a$.

142. For example, it is shown in Trigonometry that, if x stands for the arcual measure of an angle, each of the ratios

$\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ approaches indefinitely near to unity, when x is made indefinitely small. We therefore assign unity as the value of either of these ratios when $x = 0$. In this way only can we regard them as continuous functions of x when x passes through the value zero. These and similar results are expressed by using the value of the independent variable as a suffix: thus

$$\left.\frac{\sin x}{x}\right]_0 = 1 \quad \text{and} \quad \left.\frac{\tan x}{x}\right]_0 = 1.$$

143. The form $\frac{0}{0}$ may be regarded as the standard indeterminate form, but the term is also applied to functions which, on direct substitutions, take one of the form $\frac{\infty}{\infty}$, $\infty \times 0$, $\infty - \infty$, and to certain forms whose logarithms take the form $\infty \times 0$.

The term *illusory* is also applied to each of these forms because their evaluation requires some process other than the operation directly implied in the form of the function.

144. In some cases, a function in the standard indeterminate form can be evaluated by making an algebraic transformation which permits the cancelling of the factor which causes the terms to vanish for the given value of x. For example, the function

$$\frac{a - \sqrt{(a^2 - bx)}}{x}$$

takes the form $\frac{0}{0}$ when $x = 0$. Multiplying both terms by the complementary surd $a + \sqrt{(a^2 - bx)}$, we obtain

$$\frac{bx}{x[a+\sqrt{(a^2-bx}]}=\frac{b}{a+\sqrt{(a^2-bx)}}.$$

The last form is not illusory for the given value of x, since the factor which becomes zero has been removed from both terms of the fraction. The value of the fraction for $x = 0$ is therefore

$$\left.\frac{a-\sqrt{(a^2-b^2)}}{x}\right]_0=\frac{b}{2a}.$$

Evaluation by Differentiation.

145. The principles of differentiation afford us a general method by which we can derive from a function, which takes the standard indeterminate form when $x = a$, another function which, although not generally equal to the given one, has the same value when $x = a$.

For this purpose, let $\dfrac{v}{u}$ represent a fraction in which both u and v are functions of x, which vanish when $x = a$; in other words, for this value of x, we have $u = 0$ and $v = 0$.

Let P be a moving point of which the abscissa and ordinate referred to rectangular axes are simultaneous values of u and v (x not being represented in the figure); then, denoting the angle POU by θ, and the inclination of the motion of P to the axis of u by ϕ, we have

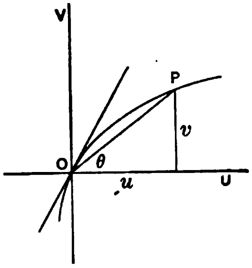

FIG. 28.

$$\tan\theta=\frac{v}{u},\qquad\text{and}\qquad\tan\phi=\frac{dv}{du}.$$

At the instant when x passes through the value a, u and v

being zero by the hypothesis, P passes through the origin; the corresponding value of θ is evidently determined by the direction in which P is moving at that instant, and is therefore equal to the value of ϕ at that point. In other words, the limiting direction of the secant OP is that of the tangent at O.

Hence the values of $\tan\theta$ and $\tan\phi$ corresponding to $x = a$ are equal, or

$$\left.\frac{v}{u}\right]_{x=a} = \left.\frac{dv}{du}\right]_{x=a};$$

therefore, to determine the value of $\frac{v}{u}$ for $x = a$, we substitute for it the function $\frac{dv}{du}$, which has the same value as the given function when $x = a$, although differing from it when x has any other value.

146. This result may also be expressed in the following manner: Let $f(x)$ and $\phi(x)$ be two functions, such that $f(a) = 0$ and $\phi(a) = 0$; then

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As an illustration, let us take $\frac{\log x}{x - 1}$. When $x = 1$, this function takes the form $\frac{0}{0}$; by the above process, we have

$$\left.\frac{\log x}{x-1}\right]_1 = \left.\frac{x^{-1}}{1}\right]_1 = 1,$$

the required value.

147. Since the substituted function $\frac{dv}{du}$ or $\frac{f'(x)}{\phi'(x)}$ frequently takes the indeterminate form, several repetitions of the process are sometimes requisite before the value of the function can be ascertained.

For example, the function $\frac{1 - \cos\theta}{\theta^2}$ takes the form $\frac{0}{0}$ when $\theta = 0$; employing the process for evaluating, we have

$$\left.\frac{1 - \cos\theta}{\theta^2}\right]_0 = \left.\frac{\sin\theta}{2\theta}\right]_0,$$

which is likewise indeterminate; but, by repeating the process, we obtain

$$\left.\frac{1 - \cos\theta}{\theta^2}\right]_0 = \left.\frac{\sin\theta}{2\theta}\right]_0 = \left.\frac{\cos\theta}{2}\right]_0 = \frac{1}{2}.$$

148. If the given function, or any of the substituted functions, contains a factor which does not take the indeterminate form, this factor may be evaluated at once, as in the following example.

The function

$$\frac{(1 - x)e^x - 1}{\tan^2 x}$$

is indeterminate for $x = 0$. By employing the usual process once, we obtain

$$\left.\frac{(1 - x)e^x - 1}{\tan^2 x}\right]_0 = \left.\frac{-xe^x}{2\sec^2 x \tan x}\right]_0,$$

which is likewise indeterminate; but, before repeating the process, we may evaluate the factor $-\dfrac{e}{2\sec^2 x}\Big]_0$. The value of this factor is — ; hence we write

$$\frac{(1-x)e^x-1}{\tan^2 x}\Big]_0 = -\frac{xe^x}{2\sec^2 x\tan x}\Big]_0 = -\frac{1}{2}\frac{x}{\tan x} = -\frac{1}{2}.$$

149. In like manner, a factor which takes the indeterminate form *but has a finite value* may be evaluated separately. Thus if the given function be

$$\frac{(e^x-1)\tan^2 x}{x^3}\Big]_0,$$

knowing that $\dfrac{\tan x}{x}\Big]_0 = 1$, we may write it in the form

$$\left(\frac{\tan x}{x}\right)^2 \cdot \frac{e^x-1}{x}\Big]_0.$$

The second factor is found on evaluation to have the value unity; hence the value of the given function is unity.

150. Again, if the given function can be separated into parts which have finite values, the parts may be evaluated separately. As an illustration, we take the expression

$$u_0 = \frac{(e^x-e^{-x})^2-2x^2(e^x+e^{-x})}{x^4}\Big]_0.$$

The process of separation into parts does not immediately apply, since the second term reduces to $-2\dfrac{e^x+e^{-x}}{x^2}$, which is infinite when $x = 0$. But, after making one application of the differential process, we have

$$u_0 = \left.\frac{2(e^x - e^{-x})(e^x + e^{-x}) - 4x(e\ + e^{-x}) - 2x^2(e^x - e^{-x})}{4x^3}\right]_0.$$

Here the last term reduces to $-\left.\frac{e^x - e^{-x}}{2x}\right]_0 = -\left.\frac{e^x + e^{-x}}{2}\right]_0 = -1.$

The rest of the expression contains the factor $e^x + e^{-x}$, which has the finite value 2 when $x = 0$. Hence the whole expression reduces to

$$u_0 = \left.\frac{e^x - e^{-x} - 2x}{x^3}\right]_0 - 1;$$

and, evaluating,

$$u_0 = \left.\frac{e^x + e^{-x} - 2}{3x^2}\right]_0 - 1 = \left.\frac{e^x - e^{-x}}{6x}\right]_0 - 1 = -\frac{2}{3}.$$

Functions which vanish with x.

151. A function of x which vanishes when $x = 0$ will generally be found to have a finite ratio to some integral power of x. Thus, if $f(x)$ is a function such that $f(0) = 0$, we have $\left.\frac{f(x)}{x}\right]_0 = f'(0)$, giving a finite value for the ratio of $f(x)$ to the first power of x, *unless* $f'(0)$ is either infinite or zero. But, if $f'(0) = 0$, we have

$$\left.\frac{f(x)}{x^2}\right]_0 = \left.\frac{f'(x)}{2x}\right]_0 = \frac{f''(0)}{2},$$

giving a finite ratio to x^2, unless the second derivative is infinite or zero. In like manner, if all the derivatives preceding $f^n(x)$ vanish when $x = 0$, while that derivative has a finite value, we shall find

$$\left.\frac{f(x)}{x^n}\right]_0 = \frac{f^n(0)}{n!}.$$

For example, each of the functions $\sin x$ and $\tan x$ vanishes with a finite ratio (namely unity) to the first power of x, because its first derivative has a finite value when $x = 0$. Again, $1 - \cos x$ vanishes with a finite ratio to x^2, because its first derivative, $\sin x$, vanishes when $x = 0$, but its second derivative does not vanish.

152. The vanishing quantities whose limiting ratios are considered above are often called *infinitesimals*, an infinitesimal being defined as *a variable whose limit is zero.* Then, taking x as the standard infinitesimal, x^2 is called *an infinitesimal of the second order.* Thus, we have seen above that, when x is infinitesimal, $\sin x$ and $\tan x$ are infinitesimals of the first order, and $1 - \cos x$ is one of the second order; in fact its ratio to x^2 is the finite quantity $\frac{1}{2}$. Again, $\tan x - \sin x$ is, under the same circumstances, an infinitesimal of the third order; for it may be written in the form $\tan x(1 - \cos x)$, which is the product of two infinitesimals of the first and second orders respectively. Accordingly,

$$\left.\frac{\tan x - \sin x}{x^3}\right]_0 = \frac{1}{2}.$$

Examples XVI.

Evaluate the following functions:

1. $\dfrac{x^3 - 5x^2 + 7x - 3}{x^3 - x^2 - 5x - 3}$, when $x = 3$. $\qquad \dfrac{1}{4}$.

2. $\dfrac{x^4 - 8x^3 + 22x^2 - 24x + 9}{x^4 - 4x^3 - 2x^2 + 12x + 9}$, when $x = 3$. $\dfrac{1}{4}$.

3. $\dfrac{\sqrt{x} - \sqrt{a} + \sqrt{(x-a)}}{\sqrt{(x^2 - a^2)}}$, $x = a$. $\dfrac{1}{\sqrt{(2a)}}$.

4. $\dfrac{x\sqrt{(3x - 2x^4)} - x^{\frac{6}{5}}}{1 - x^{\frac{2}{3}}}$, $x = 1$. $\dfrac{81}{20}$.

5. $\dfrac{(a^2 + ax + x^2)^{\frac{1}{2}} - (a^2 - ax + x^2)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}} - (a-x)^{\frac{1}{2}}}$, $x = 0$. $\sqrt{a}$.

6. $\dfrac{e^x - e^{-x}}{\log(1+x)}$, $x = 0$. 2.

7. $\dfrac{a^n - x^n}{\log a - \log x}$, $x = a$. na^n.

8. $\dfrac{xe^{2x} - e^{2x} - x + 1}{e^{2x} - 1}$, $x = 0$. -1.

9. $\dfrac{\sin x - \cos x}{\sin 2x - \cos 2x - 1}$, $x = \frac{1}{4}\pi$. $\frac{1}{2}\sqrt{2}$.

10. $\dfrac{\log x}{\sqrt{(1-x)}}$, $x = 1$. 0.

11. $\dfrac{a^x - b^x}{x}$, $x = 0$. $\log\dfrac{a}{b}$.

12. $\dfrac{\sqrt{(1+x^2)(1-x)}}{1 - x^n}$, $x = 1$. $\dfrac{\sqrt{2}}{n}$.

13. $\dfrac{a^2 - x^2}{x^2}\left(1 - \cos\dfrac{x}{a}\right)$, $x = 0$. $\dfrac{1}{2}$.

14. $\dfrac{e^{mx} - e^{ma}}{x - a}$, $x = a$. me^{ma}.

15. $\dfrac{a^{\sin x} - a}{\log \sin x}$, $x = \frac{1}{2}\pi$. $a \log a$.

16. $\dfrac{1 - \cos x}{x \log(1+x)}$, $x = 0$. $\dfrac{1}{2}$.

17. $\dfrac{\sqrt{x}\tan x}{(e^x - 1)^{\frac{3}{2}}}$, $x = 0$. 1.

18. $\dfrac{\sin x - x\cos x}{x - \sin x}$, $x = 0$. 2.

19. $\dfrac{e^x - e^{-x} - 2x}{x - \tan x}$, when $x = 0$. -1.

20. $\dfrac{(x-2)e^x + x + 2}{x(e^x - 1)^2}$, $x = 0$. $\dfrac{1}{6}$.

21. $\dfrac{x \quad - x}{1 - x + \log x}$, $x = 1$. -2.

22. $\dfrac{\tan^2 x - \sin^2 x}{x^4}$, $x = 0$. 1.

23. $\dfrac{(x-1)^2 + \sin^3 (x^2 - 1)^{\frac{1}{2}}}{(x+1)(x-1)^{\frac{3}{2}}}$, $x = 1$. $\sqrt{2}$.

24. $\dfrac{1 - x + \log x}{1 - \sqrt{(2x - x^2)}}$, $x = 1$. -1.

25. $\dfrac{\sin x - \log (e^x \cos x)}{x^2}$, $x = 0$. $\dfrac{1}{2}$.

26. $\dfrac{\frac{1}{4}\pi - \tan^{-1} x}{x^n - e^{\sin(\log x)}}$, $x = 1$. $\dfrac{1}{2(1-n)}$.

27. $\dfrac{\tan (a + x) - \tan (a - x)}{\tan^{-1}(a + x) - \tan^{-1}(a - x)}$, $x = 0$. $(1 + a^2)\sec^2 a$.

28. $\dfrac{x \sin x - \frac{1}{2}\pi}{\cos x}$, $x = \frac{1}{2}\pi$. -1.

29. $\dfrac{e^x - e^{\sin x}}{x - \sin x}$, $x = 0$. 1.

30. $\dfrac{a^{\log x} - x}{\log x}$, $x = 1$. $\log a - 1$.

31. $\dfrac{\cos^{-1}(1 - x)}{\sqrt{(2x - x^2)}}$, $x = 0$. 1.

32. $\dfrac{x^2 - a\sqrt{ax}}{\sqrt{(ax)} - a}$, $x = a$. $3a$.

33. $\dfrac{\tan nx - n \tan x}{n \sin x - \sin nx}$, $x = 0$. 2.

34. $\dfrac{\sqrt{2} - \cos x - \sin x}{\log \sin 2x}$, $x = \frac{1}{4}\pi$. $-\frac{1}{4}\sqrt{2}$.

35. $\dfrac{x + x^3 - (2n + 1)x^{2n+1} + (2n - 1)x^{2n+3}}{(1 - x^2)^2}$, $x = 1$. n^2.

36. $\dfrac{m^x \sin nx - n^x \sin mx}{\tan nx - \tan mx}$, when $x = 0$. 1.

37. $\dfrac{\tan nx - \tan mx}{\sin (n^2x - m^2x)}$, $x = 0$. $\dfrac{1}{m+n}$.

38. $\dfrac{\tan nx - \tan mx}{\sin (n^2x - m^2x)}$, $m = n$. $\dfrac{\sec^2 nx}{2n}$.

39. $\dfrac{m^x \sin nx - n^x \sin mx}{\tan nx - \tan mx}$, $m = n$.

$n^{x-1}(n \cos xn - \sin nx) \cos^2 nx$.

XVII.

The Form $\frac{\infty}{\infty}$.

153. If $f(x)$ and $\phi(x)$ are continuous functions of x both of which increase without limit when x approaches a given value a, the function $\dfrac{f(x)}{\phi(x)}$ takes the form $\frac{\infty}{\infty}$ when $x = a$. This, like $\frac{0}{0}$ is an *indeterminate form*, because its value cannot be obtained directly by division; but, since the fraction is itself a continuous function, its value is *not* indeterminate, being in fact the limit to which the ordinary values of the function approach as x approaches a.

154. It is sometimes possible to ascertain this limit by removing from both terms of the fraction by division a factor which renders them infinite. For example, the function

$$\frac{x - \sin x}{x + \cos x}$$

takes the form $\frac{\infty}{\infty}$ when $x = \infty$. But, dividing both terms by x, we have

$$\frac{x - \sin x}{x + \cos x} = \frac{1 - \dfrac{\sin x}{x}}{1 + \dfrac{\cos x}{x}}.$$

Since neither $\sin x$ nor $\cos x$ can exceed unity, $\frac{\sin x}{x}$ and $\frac{\cos x}{x}$ vanish when $x = \infty$; therefore

$$\left.\frac{x - \sin x}{x + \cos x}\right]_{\infty} = 1.$$

155. We can, in the same way, evaluate any algebraic fraction when x is infinite. For example, dividing both terms by x^5, and then making x infinite, we have

$$\left.\frac{4x^5 + 5x^4 + 3x + 10}{8x^5 - 3x^2 + 32}\right]_{\infty} = \frac{1}{2}.$$

It is obvious that, in this process, all the terms except those of the highest degree disappear from the result, which is therefore *the ratio of the terms of highest degree.* The value is finite only in case the numerator and denominator are of the same degree.

156. A fraction which takes the form ∞/∞ can be so transformed as to take the standard form $0/0$, and then treated by the differential process. For this purpose, the reciprocal of the denominator is placed in the numerator, and that of the numerator in the denominator. For example,

$$\left.\frac{\sec 3x}{\sec x}\right]_{\frac{1}{2}\pi} = \frac{\infty}{\infty} = \left.\frac{\cos x}{\cos 3x}\right]_{\frac{1}{2}\pi} = \frac{0}{0} = \left.\frac{-\sin x}{-3\sin 3x}\right]_{\frac{1}{2}\pi} = -\frac{1}{3}.$$

Since the formula for the derivative of a reciprocal involves the square of the original function, this method fails to simplify the function* unless some transformation takes place, such as that in the example above, where $\frac{1}{\sec x}$ is replaced by the simpler function $\cos x$.

Differential Formula for the Form $\frac{\infty}{\infty}$.

157. Let $\frac{v}{u}$ represent a function in which both u and v are functions of x which increase without limit as x approaches the value a. Let P be a moving point of which the abscissa and ordinate referred to rectangular axes are simultaneous values of u and v: then, denoting the angle POV by θ, we have, as in Art. 145,

$$\tan\theta = \frac{v}{u}, \qquad \tan\phi = \frac{dv}{du}.$$

In this case, as x assumes the value a, u and v by hypothesis become simultaneously infinite, and therefore the point P recedes to an infinite distance from the origin. Let OQ be the limiting direction of OP. Then, assuming that the motion of P tends to have a fixed direction (which will generally be the case),† it is obvious that the limiting direction of OP will be that of this motion; in other words,

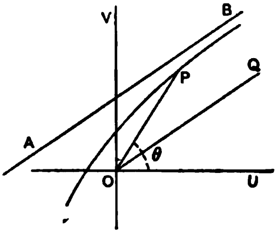

Fig. 29.

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* Thus $\frac{d}{dx}\frac{1}{f(x)} = \frac{-f'(x)}{[f(x)]^2}$. The method may, however, be used to prove the theorem geometrically demonstrated below.

† The example given in Art. 154 furnishes a case in which the motion of P does *not* tend to a fixed direction. For in that case $\tan\phi = \frac{dv}{du} = \frac{1-\cos x}{1-\sin x}$. Both

we shall have, when $x = a$, $\tan\theta = \tan\phi$. Thus

$$\left.\frac{v}{u}\right]_{x=a} = \left.\frac{dv}{du}\right]_{x=a}$$

when $\frac{v}{u}$ takes the form $\frac{\infty}{\infty}$,* as well as when it takes the form $\frac{0}{0}$.

This result, like that of Art. 145, may be otherwise expressed thus: if $\phi(a) = \infty$ and $\phi(a) = \infty$, then

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}.$$

158. Since a variable cannot become infinite in a finite interval of time while its rate is finite, a function of x *cannot become infinite, for a finite value of x, unless its derivative with respect to x is infinite.* It follows that, in the application of this formula to a case in which a is finite, the substituted function $\frac{f('x)}{\phi'(x)}$ will also take the form $\frac{\infty}{\infty}$. Hence, in order to effect the evaluation, we must be able at some point in the process to make a transformation similar to that made in Art. 156. Thus, in the example

$$\left.\frac{\log\sin 2x}{\log\sin x}\right]_0 = \frac{\infty}{\infty},$$

---

numerator and denominator vary periodically between 0 and 2, thus $\tan\phi$ is a periodic function. The curve described by P, in this case, is a prolate cycloid with its axis inclined at an angle of 45° to the axis of u.

* When, as in the diagram, the limiting ratio is a finite quantity, the curve will usually have an asymptote AB whose inclination is the limiting value of ϕ; but this is not always the case, for the distance of P from OQ may increase without limit, in which case the curve is said to have a *parabolic* branch in the direction of OQ.

by using the above formula we obtain

$$\left.\frac{\log\sin 2x}{\log\sin x}\right]_0 = \left.\frac{2\cot 2x}{\cot x}\right]_0,$$

which takes the form ∞/∞; but the last expression is equivalent to $\left.2\frac{\sin x\cos 2x}{\sin 2x\cos x}\right]_0$, and is therefore easily shown to have the value unity.

The Form $0 \times \infty$.

159. A function which takes this form may, by introducing the reciprocal of one of the factors, be so transformed as to take either of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as may be found most convenient. For example, let us take the function

$$x^{-n}e^x,$$

which, supposing n positive, assumes the above form when $x = \infty$. In this case it is necessary to reduce to the form $\frac{\infty}{\infty}$. Thus,

$$x^{-n}e^x = \left.\frac{e^x}{x^n}\right]_\infty = \left.\frac{e}{nx^{n-1}}\right]_\infty = \left.\frac{e^x}{n(n-1)x^{n-2}}\right]_\infty = \ldots$$

By continuing this process, we finally obtain a fraction whose denominator is finite while its numerator is still infinite. Hence we have, for all finite values of n,

$$x^{-n}e\,]_\infty = \infty.$$

160. When x is infinite, and $n > 1$, x^n may be called *an infinite of the nth order*, with respect to x as the standard infinite. Thus an infinite of higher order has an infinite ratio to one of lower order. The result found in Art. 159 shows that, when x is infinite, e^x is an infinite of order higher than that indicated by any finite number; that is to say, it bears an infinite ratio to every power of x.

In like manner, log x, when x is infinite, is an infinite of a *lower* order than that indicated by any positive value of n. For

$$\left.\frac{\log x}{x^n}\right]_\infty = \left.\frac{\dfrac{1}{x^{-1}}}{nx^{n-1}}\right]_\infty = \left.\frac{1}{nx^n}\right]_\infty = 0;$$

thus $\log x$ bears a zero ratio to any positive power of x, when x is infinite.

Limiting Values of Discontinuous Functions.

161. The cases which we have hitherto met in which the continuity of a function $f(x)$ is broken belong to one or the other of two kinds. In the first kind, the function becomes imaginary when x passes a particular value, say $x = a$. That is to say, $f(x)$ is imaginary for values on one side of a and real for values on the other side. The limiting value, $f(a)$, of the function is usually finite, being the value common to two values of a multiple-valued function which become equal. This is illustrated by Fig. 15, p. 70, the graph of the functions $\sin^{-1}x$; also, in the case of an implicit function, by Fig. 25, p. 130.

In cases of the second kind, the function increases without limit when x approaches a, and it is usually found, as illustrated by the graphs given in Fig. 3, p. 6, and Fig. 13, p. 64, to have positive values on one side of $x = a$ and negative values on the other. In such a case, there is no proper value of $f(a)$; but we may say that the limiting value of $f(x)$, when x approaches a from one side is $+\infty$, and when x approaches a from the other side it is $-\infty$. Thus the reciprocal $\frac{1}{x}$ is said to pass through infinity and change sign when x passes through the value zero.

162. Consider now a function of a function which becomes infinite when $x = a$. The limiting values to which the complex function approaches when x approaches a from one side or the other may be finite and equal. For example, $\tan^{-1}z$ admits of the limiting value $\frac{1}{2}\pi$ whether $z = \infty$ or $z = -\infty$; accordingly $\tan^{-1}\frac{1}{x}$ has the finite value $\frac{1}{2}\pi$ when x passes through zero. There is then no break of continuity.

On the other hand, e^z has the value ∞ when $z = +\infty$, and the value 0 when $z = -\infty$; accordingly $e^{\frac{1}{x}}$ has the limiting value ∞ when x approaches zero from the positive side, and the limiting value zero when x approaches zero from the negative side. Thus the function presents a third sort of discontinuity.* Its graph, that is the curve $y = e^{\frac{1}{x}}$, which is given in Fig. 30, is said to have a *stop-point* at the origin.

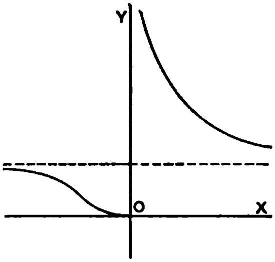

FIG. 30.

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* The function e^x may be said to have a break of continuity when $x = \infty$, while $\tan^{-1}x$ has not. Accordingly the graph of e^x, Fig. 10, p. 57, approaches the asymptote at one end only, while that of $\tan^{-1}x$, Fig. 17, p. 71, has at each end a branch approaching the asymptote.

The Form $\infty - \infty$.

163. It is obvious that the difference between two functions, each of which increases without limit as x approaches a, may approach a finite limit*: hence $\infty - \infty$ is an illusory form. A function which takes this form can be so transformed as to take the standard form. For example, each term of the function

$$\frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2}$$

takes an infinite value when $x = 0$. But, reducing to a common denominator, we have

$$\left[\frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2}\right]_0 = \left[\frac{x - (1+x)\log(1+x)}{x^2(1+x)}\right]_0 = \frac{0}{0}.$$

Hence

$$\left[\frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2}\right]_0 = \left[\frac{x - (1+x)\log(1+x)}{x^2}\right]_0$$

$$= \left[\frac{1 - \log(1+x) - 1}{2x}\right]_0 = -\frac{1}{2}.$$

Examples XVII.

Evaluate the following functions:

1. $\dfrac{\log \sec x}{\log \sec 3x}$, when $x = \frac{1}{2}\pi$. -1.

2. $\dfrac{a^x}{\operatorname{cosec}(ma^{-x})}$, $x = \infty$. m.

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* This can, of course, happen only when the *ratio* of the two quantities has unity for its limit when they become infinite.

3. $\frac{\log x}{x^n}, (n>0),$	when $x=\infty$.	0.
4. $\frac{\tan x}{\log(x-\frac{1}{2}\pi)},$	$x=\frac{1}{2}\pi$.	∞.
5. $\frac{\sec(\frac{1}{2}\pi x)}{\log(1-x)},$	$x=1$.	∞.
6. $\frac{\log\cos(\frac{1}{2}\pi x)}{\log(1-x)},$	$x=1$.	1.
7. $\frac{\tan x}{\tan 3x},$	$x=\frac{1}{2}\pi$.	3.
8. $\frac{\log(1+x)}{x},$	$x=\infty$.	0.
9. $\left(a^{\frac{1}{x}}-1\right)x,$	$x=\infty$.	$\log a$.
10. $\frac{x^2-a^2}{a^2}\tan\frac{\pi x}{2a},$	$x=a$.	$-\frac{4}{\pi}$.
11. $x^m(\log x)^n$, (*m and n being positive,*)	$x=0$.	0.
12. $e^x\sin\frac{1}{x},$	$x=\infty$.	∞.
13. $e^{-\frac{1}{x}}(1-\log x),$	$x=0$.	0.
14. $\sec\frac{\pi x}{2}.\log\frac{1}{x},$	$x=1$.	$\frac{2}{\pi}$.
15. $\frac{\log\tan nx}{\log\tan x},$	$x=0$.	1.
16. $\frac{\log\cot\frac{x}{2}}{\cot x+\log x},$	$x=0$.	0.
17. $\sec x(x\sin x-\frac{1}{2}\pi),$	$x=\frac{1}{2}\pi$.	-1.
18. $\log\left(2-\frac{x}{a}\right)\tan\frac{\pi x}{2a},$	$x=a$.	$\frac{2}{\pi}$.
19. $(1-x)\tan(\frac{1}{2}\pi x),$	$x=1$.	$\frac{2}{\pi}$.
20. $\log(x-a)\tan(x-a),$	$x=a$.	0.

21. $(a^2 - x^2)^{\frac{1}{2}} \cot \left\{ \frac{\pi}{2} \left(\frac{a-x}{a+x} \right)^{\frac{1}{2}} \right\}$, when $x = a$.

Denoting the arc by θ, *and multiplying by* $\frac{\tan \theta}{\theta}$ (*whose value, when* $x = a$, *is unity*), *we obtain* $\left. \frac{2}{\pi}(a + x) \right]_a$ $\frac{4a}{\pi}$.

22. $x^n e^{\frac{1}{x}}$, when $x = 0$. ∞ and 0.

23. $\frac{\sec^n x}{e^{\tan x}}$, $x = \frac{1}{2}\pi$. 0.

24. $x - x^2 \log\left(1 + \frac{1}{x}\right)$, $x = -\infty$. $\frac{1}{2}$.

Put $x = \frac{1}{z}$.

25. $\frac{2}{x^2 - 1} - \frac{1}{x - 1}$, $x = 1$. $-\frac{1}{2}$.

26. $\frac{\operatorname{cosec} x}{x} - \frac{\sin^{-1} x}{x^2 \sin x}$, $x = 0$. $-\frac{1}{6}$.

27. $\frac{2}{x} - \cot \frac{1}{2}x$, $x = 0$. 0.

28. $x \tan x - \frac{1}{2}\pi \sec x$, $x = \frac{1}{2}\pi$. -1.

29. $\frac{x}{x - 1} - \frac{1}{\log x}$, $x = 1$. $\frac{1}{2}$.

30. $\frac{x}{\sin^3 x} - \cot^2 x$, $x = 0$. $\frac{7}{6}$.

31. $\frac{1}{2x^2} - \frac{\pi}{2x \tan \pi x}$, $x = 0$. $\frac{\pi^2}{6}$.

32. $\frac{1}{4x} - \frac{1}{2x\,(e^{\pi x} + 1)}$, $x = 0$. $\frac{\pi}{8}$.

33. $\frac{\pi x - 1}{2x^2} + \frac{\pi}{x(e^{2\pi x} - 1)}$, $x = 0$. $\frac{\pi^2}{6}$.

34. Prove that, when $f(a) = 1$ and $\phi(a) = 1$,

$$\frac{\log f(a)}{\log \phi(a)} = \frac{f'(a)}{\phi'(a)}.$$

35. Prove that, when $f(a) = 0$ and $\phi(a) = 0$,

$$\frac{\log f(a)}{\log \phi(a)} = 1,$$

provided that $\dfrac{f'(a)}{\phi'(a)}$ is neither infinite nor zero.

XVIII.

Functions whose Logarithms Take the Form $\infty \times 0$.

164. If u and v are functions of x there are three cases in which the logarithm of the function u^v takes an illusory form. We have, in fact,

$$\log u^v = v \log u.$$

This is indeterminate when, for a particular value of x, $v = \infty$ and $\log u = 0$, in which case the given function takes the form 1^∞: it is also indeterminate when $v = 0$ and $\log u = \pm \infty$, in which case u^v takes one of the forms ∞^0 or 0^0. In each of the three cases, the function is evaluated by first evaluating its Napierian logarithm.

165. For example, the function

$$y = \left(1 + \frac{1}{x}\right)^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

takes the form 1^∞ when x increases without limit. From equation (1) we have

$$\log y = x \log \left(1 + \frac{1}{x}\right) = \frac{\log \left(1 + \frac{1}{x}\right)}{\frac{1}{x}},$$

the last expression taking the form $\frac{0}{0}$ when $x = \infty$. In evaluating it, we may conveniently put z for $\frac{1}{x}$, then

$$\log y]_\infty = \frac{\log(1+z)}{z}\Big]_0 = \frac{1}{1+z}\Big]_0 = 1.$$

Hence

$$y]_\infty = \left(1+\frac{1}{x}\right)^x\Big]_\infty = e.$$

The Napierian base is in fact often defined as the limiting value of the function in equation (1) when x increases without limit. When the graph of the function y is drawn it is the distance of an asymptote parallel to the axis of x.

166. The same function furnishes an illustration of the form ∞^0, namely when $x = 0$. Making the same substitution, $z = \frac{1}{x}$, we now have

$$\log y]_0 = \frac{\log(1+z)}{z}\Big]_\infty = \frac{1}{1+z}\Big]_\infty = 0.$$

Whence

$$y]_0 = \left(1+\frac{1}{x}\right)^x\Big]_0 = 1.^*$$

167. The function

$$y = x^x$$

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[*] This gives the point at which the graph of the function meets the axis of y. It is a stop-point since the curve is discontinuous ($\log y$ being impossible) between $x = 0$ and $x = -1$. The curve is again continuous from $x = -1$, for which y is infinite, to $x = -\infty$, for which y again approaches the limit e.

takes the form 0^0 when $x = 0$. In this case,

$$\log y\Big]_0 = x \log x\Big]_0 = \frac{\log x}{x^{-1}}\Big]_0 = -\frac{x^{-1}}{x^{-2}}\Big]_0 = 0.$$

Hence $x^x\Big]_0 = 1$.

A function which takes this form, or one which takes the reciprocal form ∞^0, is usually found to have the limiting value unity.* But this is not necessarily the case, as the expression

$$x^{\frac{a+x}{\log x}}$$

will show. This expression takes the form 0^0 when $x = 0$, but it is only another way of writing e^{a+x}; hence, when $x = 0$, its value is e^a.

Indeterminate Forms of Functions of Two Variables.

168. The value which, as explained in Art. 141, defines the ratio of two vanishing quantities is the limit of the ratio with which they vanish. Thus, in accordance with Art. 145, the value of the ratio $\dfrac{y-b}{x-a}$, when x and y simultaneously take the values a and b, is

$$\frac{y-b}{x-a}\Big]_{a,b} = \frac{dy}{dx}\Big]_{a,b};$$

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* The reason is that in either case it is the factor $\log u$ in the product $v \log u$, Art. 164, which is infinite. But when $u = 0$, as well as when $u = \infty$, as shown in Art. 160, $\log u$ bears a zero ratio to an infinite of any finite order. Hence, if v is an infinitesimal of any finite order, the product is zero, as in the example above.

that is, the value of *the vanishing ratio* (or ratio with which the terms vanish) is, in this case, the ratio of the rates with which y and x assume their special values.

169. When x and y are two independent variables, this is a *really indeterminate* quantity. In like manner, when the numerator and denominator are any functions of x and y, the value of the fraction, when it takes the indeterminate form, depends upon the relative rates of x and y at the instant of assuming the special values. For example, the function

$$\frac{x^2 - 3xy + x}{y^2 - xy + 1}$$

takes the form $\frac{0}{0}$ when $x = 2$ and $y = 1$. Employing the usual method, we obtain

$$\left.\frac{x^2 - 3xy + x}{y^2 - xy + 1}\right]_{2,1} = \left.\frac{2x - 3y + 1 - 3x\frac{dy}{dx}}{(2y - x)\frac{dy}{dx} - y}\right]_{2,1} = 6\left.\frac{dy}{dx}\right]_{2,1} - 2,$$

which is indeterminate when x and y are independent.

170. If, on the other hand, x and y are connected by an equation which admits of the simultaneous values $x = a$ and $y = b$, the value of $\left.\frac{dy}{dx}\right]_{a,b}$ (or relative rate with which x and y assume these values) is determined by the given equation. Hence, in such a case, the value of a fraction which takes the indeterminate form for these values is determinate. For example, in the case of the fraction considered in the preceding article, suppose x and y to be connected by the relation

$$x^2 + y^2 = 5y,$$

which admits of the special values (2, 1) for which the given fraction takes the indeterminate form. By differentiation, we have

$$\frac{dy}{dx} = \frac{2x}{5 - 2y}, \qquad \text{whence} \qquad \left.\frac{dy}{dx}\right]_{2,1} = \frac{4}{3}.$$

Substituting this value in the expression found above, we find the value of the fraction under the given conditions to be 6.

Since, in an example of this sort, y is an implicit function of x, the given fraction is virtually a function of the single independent variable x. We are thus able to evaluate an indeterminate form involving an implicit function.

Application to the Derivative of an Implicit Function.

171. When y is an implicit function of x its derivative presents itself in the form considered in the preceding article: that is to say, as an expression containing y as well as x. Suppose now that, for given simultaneous values of x and y, the derivative takes the indeterminate form. For example, given the equation

$$2x^2y + y^2 + 4x = 3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

from which we derive

$$\frac{dy}{dx} = -2\frac{xy + 1}{y + x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) is satisfied by the values $x = 1$, $y = -1$, and, substituting these values in equation (2), we find

$$\left.\frac{dy}{dx}\right]_{1,-1} = \frac{0}{0}.$$

Hence, applying the differential process, we have

$$\left.\frac{dy}{dx}\right]_{1,-1} = -2\left.\frac{y + x\dfrac{dy}{dx}}{\dfrac{dy}{dx} + 2x}\right]_{1,-1} = \frac{2 - 2\left.\dfrac{dy}{dx}\right]_{1,-1}}{\left.\dfrac{dy}{dx}\right]_{1,-1} + 2},$$

an equation involving the required derivative in each member. Clearing of fractions, we have the quadratic

$$\left[\frac{dy}{dx}\right]^2_{1,-1} + 4\left.\frac{dy}{dx}\right]_{1,-1} = 2,$$

whence

$$\left.\frac{dy}{dx}\right]_{1,-1} = -2 \pm \sqrt{6}.$$

Thus the derivative has two distinct values. We infer that the curve of which (1) is the equation has two branches passing through the point (1, − 1), which is therefore a *double point* of the curve. As x passes through the value 1, the implicit function y has two values which become equal when $x = 1$, but do not become imaginary. Compare Arts. 128 and 130, which treat of the cases in which one term only of the fractional value of the derivative becomes zero.

172. When $x = 0$ and $y = 0$ are simultaneous values of x and y, the equation of Art. 168 becomes

$$\left.\frac{y}{x}\right]_{0,0} = \left.\frac{dy}{dx}\right]_{0,0}$$

The curve in this case passes through the origin, and if the equation is algebraic the value of the derivative at that point may be found by evaluating the first member by an algebraic process. For example, the curve

$$x^2 + y^2 - 2x + y = 0$$

passes through the origin. Dividing the equation by x, we have

$$x + y\frac{y}{x} - 2 + \frac{y}{x} = 0.$$

Assuming $\left.\frac{y}{x}\right]_{0,0}$ to have a finite value, this equation becomes, when we put $x = 0$ and $y = 0$,

$$-2 + \left.\frac{y}{x}\right]_{0,0} = 0, \quad \therefore \quad \left.\frac{y}{x}\right]_{0,0} = 2.$$

It is obvious that in this process no terms remain, when we put $x = 0$ and $y = 0$, except those which arise from the terms of lowest degree in the original equation. Hence the result can be found by simply equating to zero the terms of lowest degree.

173. In like manner, when the equation contains no terms lower than the second in degree, the terms of the second degree determine two values of the derivative at the origin, which is then a double point. Thus the curve

$$y^3 + x^3 - 3axy = 0$$

has a double point at the origin, and the tangents at the origin are given by

$$xy = 0;$$

that is, the values of tan ϕ are 0 and ∞, and the two coordinate axes are tangents to the curve. If the values of tan ϕ are found to be imaginary there is no tangent at the origin, which is then an isolated point of the curve.

Examples XVIII.

1. $(\cos x)^{\cot^2 x}$,	when $x = 0$.	$\frac{1}{\sqrt{e}}$.
2. $\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$,	$x = 0$.	$\sqrt[3]{e}$.
3. $(\cos ax)^{\operatorname{cosec}^2 \beta x}$.	$x = 0$.	$e^{-\frac{a^2}{2\beta^2}}$.
4. $\left(\frac{1}{x}\right)^{\tan x}$,	$x = 0$.	1.
5. $(\tan x)^{\tan 2x}$,	$x = \frac{1}{4}\pi$.	$\frac{1}{e}$.
6. $\left(\frac{1}{x^n}\right)^{x^m} (m > 0)$,	$x = 0$.	1.
7. $(1 - x)^{\frac{1}{x}}$,	$x = 0$.	$\frac{1}{e}$.
8. $(\sin x)^{\sec^2 x}$,	$x = \frac{1}{2}\pi$.	$\frac{1}{\sqrt{e}}$.
9. $x^{\frac{1}{x}}$,	$x = \infty$.	1.

10. $(\sin x)^{\tan x}$, when $x = 0$. $\qquad 1$.

11. $(\sin x)^{\tan x}$, $\qquad x = \frac{\pi}{2}$. $\qquad 1$.

12. $x^{\frac{a}{\log \sin x}}$, $\qquad x = 0$. $\qquad e^a$.

13. $(\sin x)^{\frac{a^2 - x^2}{\log \tan x}}$, $\qquad x = 0$. $\qquad e^{a^2}$.

14. $x^{x^a} (a > 0)$, $\qquad x = 0$. $\qquad 1$.

15. $(x^2)^{\frac{(a + x)^2}{\log (x + \log \cos x)}}$, $\qquad x = 0$. $\qquad e^{2a^2}$.

16. $x^{\frac{1}{1-x}}$, $\qquad = 1$. $\qquad \frac{1}{e}$.

17. $x^{e^x - 1}$, $\qquad x = 0$. $\qquad 1$.

18. $(\cos mx)^{\frac{n}{x^2}}$, $\qquad x = 0$. $\qquad e^{-\frac{1}{2}nm^2}$.

19. $\left(\frac{\log x}{x}\right)^{\frac{1}{x}}$, $\qquad x = \infty$. $\qquad 1$.

20. $(1 \pm x)^{\frac{1}{x}}$, $\qquad x = \infty$. $\qquad 1$.

21. $x^m (\sin x)^{\tan x} \left(\frac{\pi - 2x}{2 \sin 2x}\right)^3$, $\qquad x = \frac{\pi}{2}$. $\qquad \frac{\pi^m}{2^{m+3}}$.

22. If $y = \frac{1}{1 + e^{\frac{1}{x}}}$, find the value of y, and also that of $\frac{dy}{dx}$, when x approaches zero from the positive and from the negative side.

$$y = 0; \ \frac{dy}{dx} = 0:$$

$$y = 1; \ \frac{dy}{dx} = 0.$$

23. If $y = \frac{x}{1 + e^{\frac{1}{x}}}$, find the value of y and of $\frac{dy}{dx}$ when x approaches zero from either side.

$$y = 0; \ \frac{dy}{dx} = 0:$$

$$y = 0; \ \frac{dy}{dx} = 1.$$

24. The variables x and y being connected by the equation

$$2(1 - x + y) - 4y^3 + 3x^4 = 0,$$

show that $x = 0$ and $y = 1$ are simultaneous values, and find the corresponding value of

$$\frac{y^3 + x^3 - 8x^2 + x - 1}{xy^2 - 4x^2}. \qquad 0.$$

25. The relation between x and y being expressed by the equation

$$3x^4 + 4y^3 - 24(x + y) + 37 = 0,$$

show that $x = 1$ and $y = 2$ are simultaneous values, and find the corresponding value of

$$\frac{y^4 - 16x}{y^4 - x^2 - 3}. \qquad 5\tfrac{9}{10}.$$

In this example, on substituting the numerical value of $\frac{dy}{dx}$, *the function again takes the indeterminate form; it is therefore necessary to substitute the value of* $\frac{dy}{dx}$ *in terms of* x *and* y, *and to repeat the process.*

26. Given $\quad x^3 - axy + a^2x - ay^2 + 2a^2y - a^3 = 0;$
show that $x = 0$ and $y = a$ are simultaneous values, and find the corresponding values of $\frac{dy}{dx}$. $\qquad$ 0 and -1.

27. Given $\quad y^4 - x^4 - 4ay^3 + 2a^2x^2 + 5a^2y^2 - 2a^3y = 0;$
show that $x = 0$ and $y = a$ are simultaneous values, and find the corresponding values of $\frac{dy}{dx}$. $\qquad \pm \sqrt{2}$.

28. Given $\quad x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0;$
find the values of $\frac{dy}{dx}$ when $y = 0$, also when $x = 0$.

$$\left.\frac{dy}{dx}\right]_{\pm a,\,0} = \pm \tfrac{2}{3}\sqrt{3}; \quad \left.\frac{dy}{dx}\right]_{0,\,-a} = \pm \tfrac{1}{3}\sqrt{6}; \quad \left.\frac{dy}{dx}\right]_{0,\,\frac{1}{2}a} = 0.$$

29. Given $\quad y^3 - a(x + a)(x + y) = 0;$
find the value of $\left.\frac{dy}{dx}\right]_{0,\,0}$. $\qquad \left.\frac{dy}{dx}\right]_{0,\,0} = -1.$

30. Given $$x^4 + ax^2y - ay^3 = 0;$$
find the values of $\frac{dy}{dx}$ when $x = 0$.

$$\left.\frac{dy}{dx}\right]_{0,\,0} = 0 \quad \text{or} \quad \pm 1.$$

31. If $$y = x(x - 1) \log (x \pm \sqrt{x}),$$
find, by the method of Art. 172, the value of $\frac{dy}{dx}$ when $x = 0$; also its values when $x = 1$, by substituting x' for $x - 1$.

$$\left.\frac{dy}{dx}\right]_{0,\,0} = \infty; \ \left.\frac{dy}{dx}\right]_{1,\,0} = -\infty \text{ or } \log 2.$$

32. Given $$y^4 + 3a^2y^2 - 4a^2xy - a^2x^2 = 0;$$
find the values of $\frac{dy}{dx}$ when $x = 0$ and $y = 0$.

$$\tfrac{1}{3}(2 \pm \sqrt{7}).$$

33. Show that the point (a, a) is an isolated point of the curve

$$x^3 - 3axy + y^3 + a^3 = 0.$$

34. Show that the point (e, e) is a double point of the curve

$$y^x = x^y.$$

CHAPTER VI.

THE DEVELOPMENT OF FUNCTIONS IN SERIES.

XIX.

Series in Ascending Powers of x.

174. A POLYNOMIAL consisting of a number of terms involving powers of x with positive integral exponents (including the exponent zero) and coefficients independent of x is called a *rational integral function* of x. Thus

$$f(x) = A + Bx + Cx^2 + \cdots + Nx^n,$$

in which A, B, C, etc., are constants, is the general type of a rational integral function. The coefficient A of the zero power is the value of the function when $x = 0$ and is called *the absolute term.*

A series of terms beginning with the absolute term A and proceeding by ascending powers of x with coefficients following a certain law (so that we can write any number of terms we choose) is called an *infinite series* in ascending powers of x. In general, for a given function of x, a series of this kind exists which is an algebraic equivalent for the function, and which will, at least for certain values of x, furnish a mode of obtaining the values of the function to any desired degree of accuracy.

The determination of this series for a given function is called *the development of the function*, and the function of which a given series is the development is called its *generating function*.

175. A rational fraction in x (that is one of which the numerator and denominator are rational and integral functions of x) affords an example in which the development can be found by an algebraic process; namely by division, the terms of dividend and divisor being arranged in ascending powers of x. For example, we thus find

$$\frac{1-x-x^2}{1+x} = 1-2x+x^2-x^3+x^4-\cdots, \quad . \quad (1)$$

where the law of the successive coefficients is that after the second they are alternately $+1$ and -1.

The algebraic equivalence of the series and the generating function, which is expressed by the sign of equality in equation (1), may be verified by multiplying both members by $1+x$. For this gives in the second member

$$\begin{array}{rrrrrl} 1 & -\,2x & +\,x^2 & -\,x^3 & +\,x^4 & -\cdots \\ & +\,x & -\,2x^2 & +\,x^3 & -\,x^4 & +\cdots \\ \hline 1 & -\,x & -\,x^2 & +\,0.x^3 & +\,0.x^4 & +\cdots \end{array}$$

which is equal to the given numerator, because we may assume the value of a series all of whose coefficients vanish to be zero for all values of x.

Convergent and Divergent Series.

176. If, in equation (1) of the preceding article, we substitute for x a numerical value, we have in the first member a value of the generating function and in the second an infinite series of numerical terms. Two very different cases may arise; these we shall illustrate by putting in turn $x = \frac{1}{2}$ and $x = 2$. In the first place, we have

$$\tfrac{1}{6} = 1 - 1 + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \cdots$$

We cannot verify this as a numerical equation, but we notice that the results of summing 3, 4, 5 and 6 terms of the series are $\frac{1}{4}$, $\frac{1}{8}$, $\frac{3}{16}$, $\frac{5}{32}$, which approach nearer and nearer to $\frac{1}{6}$ the value of the generating function. The latter is in fact the limit to which these sums approach as the number of terms included is increased indefinitely. Under these circumstances the series is said to be *convergent*, and the limit is called *the sum of the infinite series.*

177. In the second place, putting $x = 2$ we have the series

$$1 - 4 + 4 - 8 + 16 - \cdots,$$

in which the result of summing 3, 4, 5, etc., terms are values which differ from one another more and more; the series is therefore said to be *divergent.* The successive sums do not approach a limit; hence there is, in this case, no meaning to the expression "sum of the series," and no propriety in equating the series to a value of the generating function.

178. Denote the sum of the terms of the series up to and including that containing x^n by S_n, and denote by R_n the difference between S_n and the value of the generating function. Thus we shall have, for any value of x,

$$f(x) = S_n + R_n,$$

in which R_n is called *the remainder*, and depends for its value upon n as well as upon x. Then the series is convergent if

R_n tends to the limit 0 as n increases indefinitely; and it is divergent if R_n has no limit.

In the example considered in the preceding articles, the process of division gives us a general expression for the remainder; for we have

$$\frac{1 - x - x^2}{1 + x} = 1 - 2x + x^2 - x^3 + \cdots \pm x^n \mp \frac{x^{n+1}}{1 + x}.$$

Here, if x is ***numerically less than unity***, R_n decreases without limit as n increases, that is, it tends to the limit zero when $n = \infty$. Hence the series is convergent for all values of x between $+1$ and -1. These values are therefore called *the limits of convergence*.

179. In this example, the value of R_n increases without limit as n increases, for all values of x beyond the limits of convergence, and the series is, for such values, divergent. At the limits two special cases arise. When $x = 1$ the values of R_n are alternately $+\frac{1}{2}$ and $-\frac{1}{2}$. When $x = -1$, R_n is infinite for every value of n, which indicates that the generating function has an infinite value. In each case, the series not being convergent is said to be divergent.

180. Since the successive terms of the series are the differences between consecutive values of R_n, they must, in a converging series, ultimately decrease without limit as n increases. But it must not be inferred that the series is necessarily convergent when the terms so decrease.

Consider, for example, the algebraic series

$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

of which we at present suppose the generating function unknown. If $x > 1$, it is easily seen that the successive terms will ultimately increase; therefore the series is divergent. When $x = 1$, we have the numerical series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots,$$

in which the terms decrease without limit. But in this case S_n can be shown to increase without limit. For, consider the terms after the second in groups of two, four, eight, etc., ending with the terms $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, etc.; the sum of the terms in each group is greater than $\frac{1}{2}$, and the number of groups is unlimited. Therefore the series is divergent.

181. In fact, unity is for the series (1) a limit of convergence. For, supposing x positive and less than unity, every term of the series is less than the corresponding term of the series

$$x + x^2 + x^3 + x^4 + \cdots \quad . \quad . \quad . \quad . \quad (2)$$

Therefore S_n in series (1) is less than S_n in this geometrical series. But S_n in series (2) has for its limit the value $\dfrac{x}{1-x}$, which is the "sum of the series" when $x < 1$. Therefore S_n in series (1) must have a limit of less value; that is, the series is convergent when $x < 1$.

It will be noticed that we can in this way prove the convergence of any series in which, after some given term, the ratio of successive terms is always less than the common ratio in some decreasing geometrical series, that is, *less than some quantity which is itself less than unity.*

182. If the terms of a series, after a certain term, are alternately positive and negative and decrease in absolute value without limit, the series is convergent. For suppose that S_n ends with a positive term, then S_{n+1} will be less than S_n because the new negative term is greater than the new positive term. For the same reason S_{n+4} is less than S_{n+2}; hence the values of S_n, S_{n+2}, S_{n+4}, etc., decrease in magnitude. Moreover, we can show in the same manner that the values of S_{n+1}, S_{n+3}, S_{n+5}, etc , increase in magnitude. But the value of S_{n+2m}, which ends with a positive term, is greater than that of S_{n+2m+1}, which includes the next negative term. Therefore each set approaches a limit intermediate to S_n and S_{n+1}. But these limits must be the same because the difference between S_{n+2m} and S_{n+2m+1} is a term of the series, and therefore by hypothesis can be made as small as we please.

Thus, for example, the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots$$

obtained by putting $x = -1$ in the negative of series (1) of Art. 180 is convergent.

If, in the same algebraic series, x is negative and numerically greater than unity, the terms increase in numerical value and the series is divergent. Therefore $+1$ and -1 are, for this series, the limits of convergence; hence in this case we have found the series to be convergent at one of the limits and divergent at the other.

Differentiation of a Series.

183. Let us now suppose that $f(x)$ admits of development in the form

$$f(x) = A + Bx + Cx^2 + \ldots + Nx^n + R_n, \quad . \quad . \quad (1)$$

in which A, B, C, $\cdots$, N are coefficients independent of x to be determined, but R_n is an unknown function of x and also a function of n. It is assumed, however, that when $x = 0$

$R_n = 0$, no matter what the value of n. For this reason, equation (1) gives, when $x = 0$,

$$f(0) = A.$$

Thus the absolute term in the development must be the value of $f(0)$, just as it is in the rational integral function, compare Art. 174. Hence the development is impossible if $f(0)$ is infinite.* Of course the assumption that the development is possible implies, in like manner, that a finite value can be found for each of the other coefficients B, C, etc.

From the fact that R vanishes with x it follows that every series of which the generating function has a finite value when $x = 0$ must be convergent for some small values of x.

184. When the development of $f(x)$ is known that of the first derivative $f'(x)$ can be found. For example, from the known development

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad . \quad . \quad . \quad (1)$$

we obtain, by taking derivatives,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad . \quad . \quad (2)$$

These series are convergent for the same values of x, namely for values between $+1$ and -1.

185. On the other hand, if the development of $f'(x)$ is known, that of $f(x)$ can be assumed in the required form, and then the coefficients can be so determined as to make the derivative identical with the known series. Thus, supposing $f(x) = \log(1 + x)$, $f'(x) = (1 + x)^{-1}$ of which we know the development. Therefore, assuming

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* That is to say, the development is impossible in the form (1). It may happen, when $f(0)$ is infinite, that $[xf(x)]_0$ has a finite value, and that $xf(x)$ admits of development in the form (1). In that case, dividing by x we should have a development of $f(x)$ in ascending powers beginning with a term in x^{-1}.

$$\log(1 + x) = A + Bx + Cx^2 + Dx^3 + \cdots, \quad . \quad . \quad (1)$$

we have

$$\frac{1}{1 + x} = B + 2Cx + 3Dx^2 + \cdots, \quad . \quad . \quad . \quad (2)$$

which must be identical with

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots. \quad . \quad . \quad . \quad (3)$$

Hence, equating coefficients, $B = 1$, $2C = -1$, $3D = 1$, etc.; and, putting $x = 0$ in equation (1), $A = \log 1 = 0$. Therefore

$$\log(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \quad . \quad (4)$$

Changing the sign of x, we have also

$$\log(1 - x) = -(x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots).^* \quad (5)$$

Maclaurin's Theorem.

186. If we assume the development of $f(x)$, as in Art. 183, and take successive derivatives, we have the following series of developments:

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* Hence $-\log(1 - x)$ is the generating function of the series in Art. 180, which we found to be convergent for values of x between $+1$ and -1. At the limit $x = 1$ the generating function is infinite, while at the limit $x = -1$ the series is convergent.

$$f(x) = A + Bx + Cx^2 + Dx^3 + \cdots + Nx^n + R_n, \quad . \quad (1)$$

$$f'(x) = B + 2Cx + 3Dx^2 + \cdots + nNx^{n-1} + \frac{dR_n}{dx}, \quad . \quad (2)$$

$$f''(x) = 2C + 3.2Dx + \cdots + n(n-1)Nx^{n-2} + \frac{d^2R_n}{dx^2}. \quad (3)$$

$$f'''(x) = 3.2D + \cdots + n(n-1)(n-2)Nx^{n-3} + \frac{d^3R_n}{dx^3}. \quad (4)$$

Putting $x = 0$, and making the same assumption as in Art. 183, we find $A = f(0)$, $B = f'(0)$, $C = \frac{1}{2}f''(0)$, $D = \frac{1}{3.2}f'''(0)$, etc. Thus the general expression for the coefficient of x^n in equation (1) is $N = \frac{1}{n!}f^n(0)$. We infer that, if the development be possible, it is

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!}$$
$$+ f'''(0)\frac{x^3}{3!} + \cdots + f^n(0)\frac{x^n}{n!} + \cdots \quad . \quad (5)$$

This result is known as *Maclaurin's Theorem*. We shall give in the next section another demonstration, which depends upon a single differentiation, and leads also to an expression for the remainder.

187. As an example, we deduce the expansion of e^x, which is called *the exponential series*. Putting $f(x) = e^x$, we have $f'(x) = e^x$, and in general $f^n(x) = e^x$ Hence $f(0) = 1$, $f'(0) = 1$, $f^n(0) = 1$; and, substituting in Maclaurin's series,

$$e^x = 1 + x + \frac{x}{2!} + \frac{x^3}{3!} + \cdots$$

This series can be shown, by the method of Art. 181, to be convergent for all values of x, although when $x > 1$ the terms begin by increasing. When $x = 1$, we have the following numerical series for e:

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots,$$

from which the value of e may be derived with any required degree of accuracy. For example, to find e to nine decimal places the computation would be as follows:

$$\begin{array}{rr}
 & 2.5 \\
 & .16666666667 \\
 & 4166666667 \\
 & 833333333 \\
 & 138888889 \\
 & 19841270 \\
 & 2480159 \\
 & 275573 \\
 & 27557 \\
 & 2505 \\
 & 209 \\
 & 16 \\
 & 1 \\
\hline
e = & 2.71828182846
\end{array}$$

Each term is here derived from the preceding by division, and is carried to the eleventh place to insure the accuracy of the sum to the nearest unit in the ninth place.

188. Again, to expand the function sin x in powers of x, we put

$$\begin{array}{lll} f(x) = \sin x, & \therefore & f(0) = 0. \\ f'(x) = \cos x, & & f'(0) = 1. \\ f''(x) = -\sin x, & & f''(0) = 0. \\ f'''(x) = -\cos x, & & f'''(0) = -1. \\ f^{\text{iv}}(x) = \sin x, & & f^{\text{iv}}(0) = 0. \end{array}$$

Since the fourth derivative is the same as the original function, we infer that the coefficients of $\frac{x^n}{n!}$ in Maclaurin's theorem repeat the values 0, 1, 0, -1 indefinitely. Hence

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad . \quad . \quad . \quad (1)$$

In like manner, or from this equation by taking derivatives, we find

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad . \quad . \quad . \quad (2)$$

Each of these series, like the exponential series, converges for all values of x.

The Binomial Theorem.

189. The Binomial Theorem, containing $m + 1$ terms when m is a positive integer, gives the expansion of $(a + b)^m$ arranged in descending powers of a and ascending powers of b. When m is fractional or negative, by putting $x = \frac{b}{a}$ the expression becomes $a^m(1 + x)^m$, in which $(1 + x)^m$ is to be developed into an infinite series in ascending powers of x. This is readily effected by Maclaurin's Theorem. Thus, putting

$$\begin{array}{lll} f(x) = (1+x)^m, & \therefore & f(0) = 1. \\ f'(x) = m(1+x)^{m-1}, & & f'(0) = m. \\ f''(x) = m(m-1)(1+x)^{m-2}, & & f''(0) = m(m-1). \\ f'''(x) = m(m-1)(m-2)(1+x)^{m-3}, & & f'''(0) = m(m-1)(m-2). \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \end{array}$$

Hence

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

Each term of this series may be derived from the preceding one by multiplying by a factor of the form

$$\frac{m-n}{n+1}x,$$

which, when m is a positive integer, ultimately becomes zero, thus causing the series to terminate with the term containing x^m. Otherwise, the fraction ultimately becomes negative, and approaches the limit $-x$ as n increases. Therefore, by Art. 181, the series is convergent when x is between $+1$ and -1; and, by Art. 182, it is also convergent at the limit $x = 1$. At the other limit, -1, the series is divergent, the corresponding value of the generating function being infinite.

190. In the examples given in the preceding articles, we were able to write out any number of terms of the series, because the general expression for the nth derivative of the function was known. When this is not the case, the labor of finding the successive values of $f'(0)$, $f''(0)$, etc., may in some cases be abridged by expressing the successive functions of x

in terms of the preceding ones; so that, when we put $x = 0$, use may be made of numerical values already found. For example, if

$$f(x) = \tan x, \qquad f'(x) = \sec^2 x = 1 + [f(x)]^2.$$

In finding the algebraic form of subsequent derivatives, we shall, for shortness, write f, f', etc., in place of $f(x)$, $f'(x)$, etc. The whole work then stands as follows:

$$\begin{aligned}
f(x) &= \tan x, & \therefore \quad f(0) &= 0. \\
f'(x) &= 1 + f^2, & f'(0) &= 1. \\
f''(x) &= 2ff', & f''(0) &= 0. \\
f'''(x) &= 2ff'' + 2f'^2, & f'''(0) &= 2. \\
f^{\text{iv}}(x) &= 2ff''' + 6f'f'', & f^{\text{iv}}(0) &= 0. \\
f^{\text{v}}(x) &= 2ff^{\text{iv}} + 8f'f''' + 6f''^2, & f^{\text{v}}(0) &= 16. \\
f\ (x) &= 2ff^{\text{v}} + 10f'f^{\text{iv}} + 20f''f'', & f^{\text{vi}}(0) &= 0. \\
f^{\text{vii}}(x) &= 2ff^{\text{vi}} + 12f'f^{\text{v}} + 30f''f^{\text{iv}} + 20f'''^2, \\
& & f^{\text{vii}}(0) &= 12 \times 16 + 20 \times 4 = 272.
\end{aligned}$$

The process is readily continued, and substituting the results in Maclaurin's series we find

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \tfrac{62}{2835}x^9 + \cdots$$

Examples XIX.

1. Show that, for the series (2) in Art. 184,

$$R_n = \frac{(n+2)x^{n+1} - (n+1)x^{n+2}}{(1-x)^2}.$$

2. Derive an expansion by differentiating equation (2), Art. 184.

$$2(1-x)^{-3} = 1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \cdots$$

3. Expand $\log(1 - x + x^2)$.

Put $1 - x + x^2$ *in the form* $\frac{1 + x^3}{1 + x}$.

$$\log(1 - x + x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3} - \cdots$$

4. Develop $\tan^{-1}x$ by the method of Art. 185.

$$\tan {}^1x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots$$

This is known as Gregory's Series.

5. Derive the expansion of $\log(1 - x^2)$ from equation (5), Art. 185, and verify by adding the expansions of $\log(1 + x)$ and $\log(1 - x)$.

6. Show that

$$\log 2 = \frac{1}{1.2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots,$$

and that

$$1 - \log 2 = \frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6.7} + \cdots$$

7. Derive the expansion of $(1 + x)e^x$ from that of e^x.

$$(1 + x)e^x = 1 + 2x + \frac{3}{2!}x^2 + \cdots + \frac{n+1}{n!}x^n + \cdots$$

8. Derive the expansion of $(1 + x)\log(1 + x)$.

$$x + \frac{x^2}{1.2} - \frac{x^3}{2.3} + \frac{x^4}{3\cdot 4} - \cdots$$

9. Derive the expansion of $x\tan^{-1}x - \frac{1}{2}\log(1 + x^2)$.

$$\frac{x^2}{1.2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^7}{7.8} + \cdots$$

10. Find the expansion of $e^x \log(1+x)$ to the term involving x^5, by multiplying together a sufficient number of the terms of the series for e^x and for $\log(1+x)$.

$$e^x \log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots$$

Derive the following expansions:

11. $\tan^2 x = x^2 + \dfrac{2x^4}{3} + \dfrac{17x^6}{45} + \cdots$

12. $\dfrac{e^x}{1+x} = 1 + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{3x^4}{8} - \dfrac{11x^5}{30} + \cdots$

13. $\sec x = 1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \dfrac{61x^6}{6!} + \cdots$

14. $\log \sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45} + \dfrac{17x^8}{2520} + \cdots$

15. $e^x \sec x = 1 + x + x^2 + \dfrac{2x^3}{3} + \dfrac{x^4}{2} + \dfrac{3x^5}{10} + \cdots$

16. $\log(1+e^x) = \log 2 + \dfrac{x}{2} + \dfrac{x^2}{8} - \dfrac{x^4}{192} + \cdots$

17. $\sin x \cos x = x - \dfrac{2^2x^3}{3!} + \dfrac{2^4x^5}{5!} - \dfrac{2^6x^7}{7!} + \cdots$

18. $\cos^2 x = 1 - x^2 + \dfrac{2^3x^4}{4!} - \dfrac{2^5x^6}{6!} + \cdots$

19. $(1+e^x)^n = 2^n\left[1 + \dfrac{n}{2}x + \dfrac{n(n+1)x^2}{8} + \dfrac{n^2(n+3)x^3}{48} + \cdots\right].$

20. Derive the expansion of $\cos^3 x$, making use of the formula $\cos 3x = 4\cos^3 x - 3\cos x$.

$$\cos^3 x = 1 - \frac{3}{2}x^2 + \frac{3^4+3}{4.4!}x^4 - \cdots + (-1)^n \frac{3^{2n}+3}{4(2n)!}x^{2n} \cdots$$

21. Derive the expansion of $\sqrt{(1+\sin x)}$, using the formula $(\sin x + \cos x)^2 = 1 + \sin 2x$.

$$\sqrt{(1+\sin x)} = 1 + \frac{x}{2} - \frac{x^2}{2^2.2!} - \frac{x^3}{2^3.3!} + \frac{x^4}{2^4.4!} + \cdots$$

22. Prove the Binomial Theorem for negative integers by successive differentiation of the series for $(1+x)^{-1}$.

XX.

Taylor's Theorem.

191. Maclaurin's Theorem gives a development of the difference, $f(x) - f(0)$, between the values of a function f corresponding to two values, namely 0 and x, of the independent variable. Of these values 0 may be called the *initial* and x the *final* value of the independent variable. The quantity whose powers appear in the development is the difference between these values, and the coefficients of the powers were found to depend upon the initial value.

We shall now show that the difference, $f(x_1) - f(x_0)$, between any two values of the function may in like manner be developed in powers of the difference, $x_1 - x_0$, between the values of the independent variable, with coefficients depending upon the initial value x_0 and independent of the final value x_1.

192. If we denote the difference $x_1 - x_0$ by h, so that

$$x_1 = x_0 + h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the development is required of $f(x_0 + h)$ in powers of h and coefficients depending upon x_0, that is to say in the form

$$f(x_0 + h) = f(x_0) + B_0 h + C_0 h^2 + \cdot \cdot \cdot + N_0 h^n + R_0. \quad (2)$$

In this equation the coefficients are marked with the suffix zero to show that they are functions of x_0. R_0, the remainder after the term containing h^n, is so marked for the same reason, and it is also to be noticed that the assumed form of equation (2) makes R_0 *vanish with h.**

Substituting the value of h, we may write equation (2) in the form

$$f(x_1) = f(x_0) + B_0(x_1 - x_0) + C_0(x_1 - x_0)^2 \\ + D_0(x_1 - x_0)^3 + \cdots + N_0(x_1 - x)^n + R_0. \quad (3)$$

If in this equation x_0 be made to vary while x_1 is constant, the quantities $B_0, \cdots, N_0$ and R_0 will be functions of x_0, and their forms may be ascertained by differentiating the equation on this hypothesis. In doing this, it will be convenient to replace x_0 by x, thus writing

$$f(x_1) = f(x) + B(x_1 - x) + C(x_1 - x)^2 \cdots + N(x_1 - x)^n + R, \quad (4)$$

in which $B, \cdots, N$ and R are functions of x, while in equation (3) $B_0, \cdots, N_0$ and R_0 are the special values which they take when $x = x_0$.

193. Taking derivatives with respect to x, equation (4) gives

$$0 = f'(x) - B + (x_1 - x)\frac{dB}{dx} - 2C(x_1 - x) + (x_1 - x)^2\frac{dC}{dx} \cdots \\ - nN(x_1 - x)^{n-1} + (x_1 - x)^n\frac{dN}{dx} + \frac{dR}{dx}. \quad \cdot \quad \cdot \quad (5)$$

To render the development possible, $B, C, \cdots, N$ and R

---

* Compare Art. 183, in which the assumption that R vanishes with x is shown to be equivalent to making the absolute term identical with the initial value of the function.

must have such values as will make equation (5) ***identical***, that is to say, true for all values of x.

It is evident that B may be so taken as to cause the first two terms of equation (5) to vanish, and that, this being done, C can be so determined as to cause the coefficient of $(x_1 - x)$ to vanish, D so as to make the coefficient of $(x_1 - x)^2$ vanish, and so on. The requisite conditions are

$$f'(x) - B = 0, \quad \frac{dB}{dx} - 2C = 0, \quad \frac{dC}{dx} - 3D = 0, \text{ etc.,} \quad (6)$$

and finally

$$(x_1 - x)^n \frac{dN}{dx} + \frac{dR}{dx} = 0. \quad . \quad . \quad . \quad . \quad (7)$$

From conditions (6) we derive

$$B = f'(x), \qquad C = \frac{1}{2}\frac{dB}{dx} = \frac{1}{2}f''(x),$$

$$D = \frac{1}{3}\frac{dC}{dx} = \frac{1}{3!}f'''(x), \qquad E = \frac{1}{4!}f^{\text{IV}}(x),$$

. . . .

and in general

$$N = \frac{1}{n!}f^n(x .$$

Putting x_0 for x, and substituting in equation (2) the values of A_0, B_0, C_0, $\cdot \cdot \cdot$, N_0, we obtain

$$f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2!} + \cdots + f^n(x_0)\frac{h}{n!} + R_0. \quad (8)$$

This result is called Taylor's Theorem, from the name of its discoverer, Dr. Brook Taylor, who first published it in 1715.

It is evident from equation (8) that *the proposed expansion is impossible* when the given function or any of its derived functions is infinite for the value x_0.

Expressions for the Remainder.

194. To completely satisfy equation (5) as an identity, R must be such a function of x as to satisfy equation (7), which, after substituting the value found for N, becomes

$$\frac{dR}{dx} = -\frac{(x_1 - x)^n}{n!} f^{n+1}(x). \qquad (9)$$

We likewise have the condition, Art. 192, that $R = 0$ when $h = 0$, that is when $x = x_1$. R_0 is then the value of R corresponding to $x = x_0$.

It follows that, in order to have a definite value of R_0, it is necessary that R should be a continuous function of x for the range of values between x_1 and x_0. This requires that its derivative in equation (9) should be finite and continuous for the same range. Hence it is necessary to the existence of equation (8) that $f^{n+1}(x)$ should not become infinite or imaginary for any value of x between x_0 and x_1. Since this implies that the preceding derivatives of $f(x)$ are likewise continuous for the same range, we may state the necessary condition to be that $f(x)$ *and all its derivatives to the* $(n + 1)$*th inclusive shall remain finite and real while* x *varies from* x_0 *to* $x_0 + h$.

195. Assuming this condition to be fulfilled, various expressions for the remainder can be found. These expressions, although containing an undetermined quantity, may serve to restrict the numerical value of R_0 between certain limits.

For this purpose, we assume a function of x which varies continuously from the value unity to the value zero, while x varies from x_0 to x_1. For example,

$$\frac{(x_1 - x)^{n+1}}{h^{n+1}}$$

is such a function. Multiplying by R_0, we have the function

$$P = \frac{(x_1 - x)^{n+1}}{h^{n+1}} R_0, \quad . \quad . \quad . \quad . \quad (10)$$

which has, for each of the extreme values, x_0 and x_1, the same value as R, namely R_0 and zero respectively. It follows that $P - R$ is a continuous function of x which has the value zero for each of the extreme values of x. Hence, as x varies from x to x_1, $P - R$ starting from the value zero and returning to that value must pass through at least one value which is numerically a maximum. Therefore the derivative of $P - R$ will take the value zero for at least one value of x between x_0 and x_1.

Since $x_1 = x_0 + h$, such an intermediate value of x may be denoted by $x_0 + \theta h$, where θ is a *positive proper fraction.* Hence we may put

$$\left.\frac{dP}{dx}\right]_{x_0+\theta h} = \left.\frac{dR}{dx}\right]_{x_0+\theta h}.$$

Using the value of P assumed in equation (10), we have

$$\frac{dP}{dx} = -\frac{(n+1)(x_1 - x)^n}{h^{n+1}} R_0.$$

Substituting $x_0 + \theta h$ for x in this and in the value of $\frac{dR}{dx}$, equation (9), we have, on equating the results,

$$R_0 = \frac{h^{n+1}}{(n+1)!} f^{n+1}(x_0 + \theta h).^*$$

This expression for the remainder was first given by Lagrange. Employing it, equation (8) becomes

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2!} + \cdots$$
$$+ f^n(x_0)\frac{h^n}{n!} + f^{n+1}(x_0 + \theta h)\frac{h^{n+1}}{(n+1)!}. \quad . \quad . \quad (11)$$

It will be noticed that Lagrange's expression for the remainder after $n + 1$ terms differs from the next term of the series, simply by the addition of θh to x_0.

---

* Other forms of the remainder result from assuming P in other forms. For example,

$$P = \frac{f^n(x_1) - f^n(x)}{f^n(x_1) - f^n(x_0)} R_0$$

satisfies the necessary conditions, and results in

$$R_0 = (1 - \theta)^n [f^n(x_0 + h) - f^n(x_0)] \frac{h^n}{n!}.$$

This value of R_0 lies between

$$0 \qquad \text{and} \qquad [f^n(x_0 + h) - f^n(x_0)]\frac{h^n}{n!};$$

therefore, see equation (8), $f(x_0 + h)$ lies between the two expressions

$$f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2!} + \cdots + f^n(x_0)\frac{h^n}{n!}$$

and

$$f(x_0) + f'(x_0)h + f''(x_0\frac{h^2}{2!} + \cdots + f^n(x_0 + h\ \frac{h^n}{n!}.$$

See Messenger of Mathematics, New Series, vol. ii, p. 180.

196. The condition given in Art. 194, although *necessary*, is not a *sufficient* one for the convergence of the series.* But, if the series is convergent, the expression for R may be used to determine a limit to the error committed in taking S_n for the sum of the series.

For example, Maclaurin's Theorem is the result of putting $x_0 = 0$ and $h = x$ in Taylor's Theorem. Hence in the exponential series, Art. 187, the remainder after the term $\frac{x^n}{n!}$ is, by equation (11), $e^{\theta x}\frac{x^{n+1}}{(n+1)!}$ Thus, in the numerical computation given on page 181, in which $x = 1$ and $n = 14$, the remainder after the term $\frac{1}{14!}$ is $\frac{e^\theta}{15!}$, which (because e is less than 3) is less than $\frac{1}{5}$ of the last term. It is therefore far too small to affect the result.

Computation by Numerical Series.

197. For a given form of the function f, Maclaurin's Theorem is the special case of Taylor's Theorem in which the initial value is zero. But any development which can be made by Taylor's can also, by a change of the form of f, be made by Maclaurin's Theorem. For example, if $\log(1 + h)$ is to be developed by Taylor's Theorem, the symbol f is given the meaning log, that is $f(x) = \log x$, and $x_0 = 1$. But, if we change the form of the function, and write $F(x) = \log(1+x)$, we obtain the same development from Maclaurin's Theorem (x taking the place of h), while the coefficients, which were before represented by $f(1)$, $f'(1)$, etc., are now represented by $F(0)$, $F'(0)$, etc. Their values would be found thus:

---

* The convergence, as we have seen in the preceding section, depends upon the character of R considered as a function of n.

$$
\begin{aligned}
F(x) &= \log(1+x), & \therefore\quad F(0) &= 0.\\
F'(x) &= \frac{1}{1+x}, & F'(0) &= 1.\\
F''(x) &= -(1+x)^{-2}, & F''(0) &= -1.\\
F'''(x) &= 2(1+x)^{-3}, & F'''(0) &= 2.\\
F^{\text{iv}}(x) &= -2.3(1+x)^{-4}, & F^{\text{iv}}(0) &= -2.3.\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & &\cdot\ \cdot\ \cdot\ \cdot
\end{aligned}
$$

Whence, substituting in equation (5), Art. 186, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad . \quad . \quad (1)$$

which is known as *the logarithmic series*, and has already been otherwise derived in Art. 185.

198. This series is divergent for values of x greater than unity, and is very slowly convergent except for very small values of x. For the practical computation of Napierian logarithms, a series for the difference of two logarithms has been deduced, which may be employed in computing successive logarithms; that is, denoting the numbers corresponding to two logarithms by n and $n+h$, we require a series for

$$\log(n+h) - \log n = \log\frac{n+h}{n}.$$

A series which could be employed for this purpose might be obtained from equation (1) by putting $\frac{n+h}{n}$ in the form

$1+\frac{h}{n}$. We obtain, however, a much more rapidly converging series by the process given below.

Substituting $-x$ for x in equation (1), we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad . \quad . \quad (2)$$

Subtracting from equation (1),

$$\log\frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right], \quad . \quad (3)$$

a series involving only the positive terms of series (1).

Putting $\frac{1+x}{1-x} = \frac{n+h}{n}$, we derive $x = \frac{h}{2n+h}$; substituting in equation (3), we have

$$\log\frac{n+h}{n} = 2\left[\frac{h}{2n+h} + \frac{1}{3}\frac{h^3}{(2n+h)^3} + \frac{1}{5}\frac{h^5}{(2n+h)^5} + \cdots\right]. \quad (4)$$

199. Suppose, for example, it is required to compute log 2, it would be quite impracticable to use for the purpose the result of putting $x=1$ in equation (1), owing to the alternate signs, and extremly slow convergence of the series. But, if we put $n=1$ and $h=1$ in equation (4), we have

$$\log 2 = 2\left[\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \frac{1}{7}\cdot\frac{1}{3^7} + \cdots\right],$$

a series which converges with considerable rapidity.

In making the computation, it is convenient first to obtain the values of the powers of $\frac{1}{3}$ which occur in the series for log 2, by successive division by 9, and afterwards to derive the values of the required terms of the series by dividing these auxiliary numbers by 1, 3, 5, 7, etc. Thus:

$\frac{1}{3}$	0.3333333333 :	1	0.3333333333
$(\frac{1}{3})^3$	370370370 :	3	123456790
$(\frac{1}{3})^5$	41152263 :	5	8230453
$(\frac{1}{3})^7$	4572474 :	7	653211
$(\frac{1}{3})^9$	508053 :	9	56450
$(\frac{1}{3})^{11}$	56450 :	11	5132
$(\frac{1}{3})^{13}$	6272 :	13	482
$(\frac{1}{3})^{15}$	697 :	15	46
$(\frac{1}{3})^{17}$	77 :	17	5
			0.3465735902
			2
			$\log_e 2 = 0.69314718^{04}$

200. The Napierian logarithms of the successive natural numbers might thus be computed by giving to n the successive values 2, 3, 4, etc., and retaining $h = 1$. But more convenient series are obtained, in some cases, by employing other values.

Thus for $\log_e 10$, if we put $n = 8$ and $h = 2$, we have, since $\log 8 = 3 \log 2$,

$$\log_e 10 = 3 \log_e 2 + \frac{2}{3}\left[\frac{1}{3} + \frac{1}{3}\,\frac{1}{3^5} + \frac{1}{5}\,\frac{1}{3^9} + \frac{1}{7}\,\frac{1}{3^{13}} + \cdots\right].$$

The same auxiliary numbers occur as in the computation of log 2 above; thus we have:

$\frac{1}{3}$	0.3333333333 : 1	0.3333333333
$(\frac{1}{3})^5$	41152263 : 3	13717421
$(\frac{1}{3})^9$	508053 : 5	101611
$(\frac{1}{3})^{13}$	6272 : 7	896
$(\frac{1}{3})^{17}$	77 : 9	9
		3)0.3347153270
		0.1115717757
		0.2231435513
	$3 \log_e 2 =$	2.0794415412
	$\log_e 10 =$	2.302585 09

201. The common or tabular logarithms, of which 10 is the base, are derived from the corresponding Napierian logarithms by means of the relation

$$\log_e x = \log_e 10 \log_{10} x,$$

whence

$$\log_{10} x = \frac{1}{\log_e 10} \log_e x = M \log_e x.$$

The constant $\frac{1}{\log_e 10}$, denoted above by M, is called the *modulus* of common logarithms. Taking the reciprocal of $\log_e 10$, computed above, we have

$$M = 0.43429448.$$

Application to Maxima and Minima.

202. If the initial value x_0 is one of the critical values, Art. 111, for the function f, we have $f'(x_0) = 0$, and the series for the difference $f(x_0 + h) - f(x_0)$ reduces to

$$f''(x_0)\frac{h^2}{2!}+f'''(x_0)\frac{h^3}{3!}+\cdots$$

Supposing $f''(x_0)$ to have a finite value, it is obvious that there will exist a value of h so small that for all smaller values the value of the first term of this expression will be numerically greater than the sum of all the others, so that the sign of the expression will be that of its first term. In other words, it will be that of $f''(x_0)$, whether h is positive or negative. Accordingly, if $f''(x_0)$ has a negative value, the values of $f(x_0+h)$ will be less than that of $f(x_0)$ for values in the immediate neighborhood of x_0. Thus $f(x_0)$ will be a maximum in accordance with Art. 116. In like manner, a minimum is indicated by a positive value of $f''(x_0)$.

203. But, if $f''(x_0)$ as well as $f'(x_0)$ vanishes, while $f'''(x_0)$ does not, the expression for the difference $f(x_0+h)-f(x_0)$ will begin with the term containing h^3; and, in this case, its sign, which for small values of h is the sign of its first term, will change with the sign of h. Thus the neighboring values to $f(x_0)$ will exceed $f(x_0)$ on one side and fall below it on the other; so that $f(x_0)$ is neither a maximum nor a minimum, in accordance with Art. 117.

Again, if $f'''(x_0)$ also vanishes, but $f^{\text{iv}}(x_0)$ does not, the difference begins with the term containing h^4, and for small values of h does not change sign with h; so that the same conclusions follow as in the case when $f''(x_0)$ is the lowest derivative which does not vanish, compare Arts. 118 and 119.

Evaluation of Vanishing Fractions by Development.

204. A function which vanishes with x becomes, when developed by Maclaurin's Theorem, a series beginning with the term containing x or a higher power of x. Denoting this power by x^n, the function may be expressed as the product of x^n by a series of which the absolute term is the original coefficient of x^n. We thus ascertain at once the power of x to which the function bears a finite ratio as it vanishes and the value of that ratio. Compare Art. 151, in which it will be noticed that the value found for this ratio agrees with the coefficient of x^n in Maclaurin's Theorem.

205. When both terms of the fraction $\dfrac{f(x)}{\phi(x)}$ vanish with x, the fraction will have a finite value only when the development of each term begins with the same power of x. Thus, the vanishing fraction

$$\left.\frac{x\sin(\sin x) - \sin^2 x}{x^6}\right]_0$$

will have an infinite value if the numerator is found, on development, to contain a term lower in degree than x^6, and the value will be zero if it contains no term lower than x^7. It is therefore unnecessary, in this case, to retain in the development of the numerator any term whose degree is higher than the sixth; and hence, in that of sin (sin x), no terms need be retained higher in degree than x^5. Employing the series for sin x, Art. 188, we have

$$\begin{aligned}\sin(\sin x) &= \sin x - \tfrac{1}{6}\sin^3 x + \tfrac{1}{120}\sin^5 x - \cdots \\ &= (x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 \cdots) - \tfrac{1}{6}(x^3 - \tfrac{1}{2}x^5 \cdots) + \tfrac{1}{120}x^5 \cdots \\ &= x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5 \cdots,\end{aligned}$$

whence

$$x\sin(\sin x) = x^2 - \tfrac{1}{3}x^4 + \tfrac{1}{10}x^6 \cdots;$$

also, squaring the series for sin x,

$$\sin^2 x = x^2 - \tfrac{1}{3}x^4 + \tfrac{2}{45}x^6 \cdots;$$

hence

$$x \sin(\sin x) - \sin^2 x = \tfrac{1}{18}x^6 \ldots$$

The value of the given fraction is therefore $\frac{1}{18}$.

Examples XX.

1. Expand e^{x_0+h} by Taylor's Theorem, and thence show that $e^{x_0+h} = e^{x_0}e^h$.

2. Expand $\log(x_0 + h)$, writing the general term and the expression for the remainder.

$$\log(x_0 + h) = \log x_0 + \frac{h}{x_0} - \frac{h^2}{2x_0^2} + \frac{h^3}{3x_0^3} - \frac{h^4}{4x_0^4} + \cdots$$

$$-(-1)^n\frac{h^n}{nx_0^n} + (-1)^n\frac{h^{n+1}}{(n+1)(x_0+\theta h)^{n+1}}.$$

3. Find the expansion of $f(x_0 + h)$, when $f(x) = x \log x - x$, writing the $(n + 1)$th term of the series.

$$f(x_0 + h) = x_0 \log x_0 - x_0 + \log x_0 \,.\, h + \frac{1}{x_0}\cdot\frac{h^2}{1.2} - \frac{1}{x_0}\cdot\frac{h^3}{2.3} + \cdots$$

$$+ (-1)^n\frac{1}{x_0^{n-1}}\cdot\frac{h^n}{(n-1)n} \cdots$$

4. Prove that

$$\sin(\tfrac{1}{6}\pi + h) = \frac{1}{2}\Big] 1 + h\sqrt{3} - \frac{h^2}{2!} - \frac{h^3\sqrt{3}}{3!} + \frac{h^4}{4!} + \frac{h^5\sqrt{3}}{5!} - \cdots\Big].$$

5. Prove that

$$\tan(\tfrac{1}{4}\pi + h) = 1 + 2h + 2h^2 + \tfrac{8}{3}h^3 + \tfrac{10}{3}h^4 + \cdots$$

6. Compute $\log_e 3$, and find $\log_{10} 3$ by multiplying by the value of M (Art. 201).

$\log_e 3 = 1.0986123.$

$\log_{10} 3 = 0.4771213.$

7. Find $\log_e 269$.

Put $n = 270 = 10 \times 3^3$, *and* $h = -1$.

$\log_e 269 = 5.5947114$.

8. Find $\log_e 7$, and $\log_e 13$. $\log_e 7 = 1.9459101$.
$\log_e 13 = 2.5649494$.

Evaluate the following functions by the method of Art. 205:

9. $\dfrac{m \sin \theta - \sin m\theta}{\theta(\cos \theta - \cos m\theta)}$, when $\theta = 0$. $\dfrac{m}{3}$.

10. $\dfrac{1}{x} - \dfrac{(1 + x) \log (1 + x)}{x^2}$, $x = 0$. $-\dfrac{1}{2}$.

11. $\dfrac{(x + \sin 2x - 6 \sin \frac{1}{2}x)^2}{(4 + \cos x - 5 \cos \frac{1}{2}x)^3}$, $x = 0$. $\dfrac{8.29^2}{3^2}$.

12. $\dfrac{\tan \pi x - \pi x}{2x^2 \tan \pi x}$, $x = 0$. $\dfrac{\pi^2}{6}$.

13. $\dfrac{\theta(2\theta + \sin 2\theta - 4 \sin \theta)}{3 + \cos 2\theta - 4 \cos\theta}$, $\theta = 0$. $-\dfrac{4}{3}$.

XXI.

The General Term of the Development.

206. Maclaurin's Theorem enables us to write the general term of the development of a function when the expression for the nth derivative is known, as in the simple cases of the series for e^x, $\sin x$, $\cos x$, $\log (1 + x)$ etc. Again, putting $x = 0$ in equation (4), Art. 103, we have for the coefficient of $\dfrac{x^n}{n!}$ in the development of $e^{ax} \cos bx$

$$(a^2 + b^2)^{\frac{1}{2}n} \cos \left(n \tan^{-1}\frac{b}{a}\right).$$

In the special case where $a = b = 1$, we can assign the numerical values, since this expression reduces then to

$$(\sqrt{2})^n \cos n\frac{\pi}{4},$$

which is therefore the coefficient of $\frac{x^n}{n!}$ in the development of $e \cos x$. The cosine takes periodically the eight successive values $\sqrt{\frac{1}{2}}$, 0, $-\sqrt{\frac{1}{2}}$, -1, $-\sqrt{\frac{1}{2}}$, 0, $\sqrt{\frac{1}{2}}$, 1. The law of the coefficients is best seen by separating the terms of even and odd degree; thus

$$e^x \cos x = \begin{cases} 1 - 4\frac{x^4}{4!} + 16\frac{x^8}{8!} - 64\frac{x^{12}}{12!} + \cdots \\ + x - 2\frac{x^3}{3!} - 4\frac{x^5}{5!} + 8\frac{x^7}{7!} + \cdots \end{cases}$$

Employment of Differential Equations.

207. When the general expression for the nth derivative cannot be obtained, it may yet be possible to find an expression for the particular value which it assumes when $x = 0$. This is done by establishing a linear relation between the values of successive derivatives, as illustrated below:

Let it be required to develop $\sin mz$ in integral powers of $\sin z$; or, what is the same thing, putting $\sin z = x$, to develop $\sin [m \sin^{-1} x]$ in powers of x. Putting

$$y = \sin [m \sin^{-1} x], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we have

$$\frac{dy}{dx} = \frac{m \cos [m \sin^{-1} x]}{\sqrt{(1 - x^2)}}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\frac{d^2y}{dx^2} = \frac{-m^2 \sin\left[m \sin^{-1}x\right] + m \cos\left[m \sin^{-1}x\right] \dfrac{x}{\sqrt{(1-x^2)}}}{1-x^2}. \qquad (3)$$

Substituting from equations (1) and (2) in equation (3), we have

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0, \quad . \quad . \quad . \quad (4)$$

a linear relation between y and its derivatives.

Taking the nth derivative of each term of this equation, by means of Leibnitz's theorem, Art. 105, we find

$$\left.\begin{aligned} (1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2nx\frac{d^{n+1}y}{dx^{n+1}} - n(n-1)\frac{d^ny}{dx^n} \\ - x\frac{d^{n+1}y}{dx^{n+1}} - n\frac{d^ny}{dx^n} \\ + m^2\frac{d^ny}{dx^n} \end{aligned}\right\} = 0,$$

or

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2-n^2)\frac{d^ny}{dx^n} = 0, \quad . \quad . \quad (5)$$

a linear relation between any three successive derivatives of the given function.

208. When $x = 0$, this relation takes the simpler form

$$\left.\frac{d^{n+2}y}{dx^{n+2}}\right]_0 = (n^2-m^2)\left.\frac{d^ny}{dx^n}\right]_0. \quad . \quad . \quad . \quad . \quad (6)$$

Now, from equations (2) and (3), we obtain

$$\left.\frac{dy}{dx}\right]_0 = m, \qquad \text{and} \qquad \left.\frac{d^2y}{dx^2}\right]_0 = 0.$$

Hence, putting n equal to 1, 3, 5 etc. in equation (6), we have

$$\left.\frac{d^3y}{dx^3}\right]_0 = m(1 - m^2), \qquad \left.\frac{d^5y}{dx^5}\right]_0 = m(1 - m^2)(9 - m^2) \text{ etc.},$$

and, for all even values of n,

$$\left.\frac{d^ny}{dx^n}\right]_0 = 0.$$

Substituting these values in Maclaurin's Theorem, we have [since $f(0) = 0$]

$$\sin(m \sin^{-1}x) = mx + \frac{m(1 - m^2)}{3!}x^3 + \frac{m(1 - m^2)(9 - m^2)}{5!}x^5 + \cdots; \quad (7)$$

or, replacing x by $\sin z$,

$$\sin mz = m \sin z\left[1 - \frac{m^2 - 1}{3!}\sin^2 z + \frac{(m^2 - 1)(m^2 - 9)}{5!}\sin^4 z - \cdots\right] \quad (8)$$

This series will consist of a finite number of terms when m is an *odd* integer.

In a similar manner, it may be proved that

$$\cos mz = 1 - \frac{m^2}{2!}\sin^2 z + \frac{m^2(m^2 - 4)}{4!}\sin^4 z - \cdots, \quad (9)$$

the number of terms being finite when m is an even integer.

209. As another example of a function satisfying a linear differential equation,* let

$$y = (\sin^{-1}x)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

---

* A direct method of finding the development of the function satisfying a given linear differential equation will be found in Differential Equations, p. 166 *et seq.*; see also Higher Mathematics, John Wiley & Sons, p. 344.

then

$$\frac{dy}{dx} = \frac{2\sin^{-1}x}{\sqrt{(1-x^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\frac{d^2y}{dx^2} = 2\frac{1 + \dfrac{x\sin^{-1}x}{\sqrt{(1-x^2)}}}{1-x^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Combining equations (2) and (3), we obtain the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Taking, by means of Leibnitz' theorem, the *n*th derivative of each term, we have

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2nx\frac{d^{n+1}y}{dx^{n+1}} - n(n-1)\frac{d^ny}{dx^n} - x\frac{d^{n+1}y}{dx^{n+1}} - n\frac{d^ny}{dx^n} = 0,$$

whence, putting $x = 0$, we derive

$$\left.\frac{d^{n+2}y}{dx^{n+2}}\right]_0 = n^2\left.\frac{d^ny}{dx^n}\right]_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From equations (2) and (3) we have

$$\left.\frac{dy}{dx}\right]_0 = 0, \qquad \left.\frac{d^2y}{dx^2}\right]_0 = 2.$$

Hence, from equation (5), the odd derivatives vanish, and for the even derivatives we have

$$\left.\frac{d^2y}{dx^2}\right]_0 = 2, \qquad \left.\frac{d^4y}{dx^4}\right]_0 = 2.2^2, \qquad \left.\frac{d^6y}{dx^6}\right]_0 = 2.2^2.4^2, \qquad \text{etc.}$$

Finally, substituting in Maclaurin's Theorem, we obtain the expansion

$$(\sin^{-1}x)^2 = 2\left[\frac{x^2}{1.2} + \frac{2^2x^4}{1.2.3.4} + \frac{2^2.4^2x^6}{1.2\ldots6} + \frac{2^2.4^2.6^2x^8}{1.2\ldots8} + \cdots\right],$$

which may be written in the form

$$(\sin^{-1}x)^2 = 2\left[\frac{x^2}{2} + \frac{2}{3}\frac{x^4}{4} + \frac{2.4}{3.5}\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\frac{x^8}{8} + \cdots\right]. \qquad (6)$$

210. Differentiation of the result obtained above gives the development

$$\frac{\sin^{-1}x}{\sqrt{(1 - x^2)}} = x + \frac{2}{3}x^3 + \frac{2.4}{3.5}x^5 + \frac{2.4.6}{3.5.7}x^7 + \cdots$$

This is equivalent to a development of the arcual measure of an angle in the form of a series involving powers of its sine; for, putting $\sin^{-1}x = \theta$, we have

$$\theta = \cos\theta \sin\theta\left[1 + \frac{2}{3}\sin^2\theta + \frac{2.4}{3.5}\sin^4\theta + \cdots\right].$$

Again, if we further transform this by putting x for $\tan\theta$, so that $\sin^2\theta = \dfrac{x^2}{1 + x^2}$, we have

$$\tan^{-1}x = \frac{x}{1 + x^2}\left[1 + \frac{2}{3}\frac{x^2}{1 + x^2} + \frac{2.4}{3.5}\left(\frac{x^2}{1 + x^2}\right)^2 + \cdots\right]. \qquad (1)$$

This series, which was given by Euler in 1793, is convergent for all values of x.

Computation of π.

211. Since $\tan^{-1}1 = \frac{1}{4}\pi$, a series for computing π may be obtained by putting $x = 1$ in the equation just found, thus

$$\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots,$$

but this series converges very slowly.

The earliest computations of π by series were made with the aid of Gregory's Series, Ex. XIX. 4, p. 185:

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots$$

In this way, Abraham Sharp in 1699 computed π to 71 places, using $x = \frac{1}{3}\sqrt{3}$, whence $\tan^{-1}x = \frac{1}{6}\pi$. But, in general, much smaller values of x were made available by using trigonometric formulæ in which $\frac{1}{4}\pi$ is separated into two or more smaller arcs or multiples of such arcs. Thus, Euler employed

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3},$$

and Machin computed π to 100 places, using

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

Again, Hutton in 1776 employed

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}.$$

212. When Euler's series, equation (1), Art. 210, is used, those values of x which give to the fraction $\dfrac{x^2}{1 + x^2}$ a decimal denominator are particularly convenient. Thus, using the formula

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79},^{*} \quad . \; . \; . \; . \; . \quad (1)$$

for $x = \frac{1}{7}$, the fraction

$$\frac{x^2}{1+x^2} = \frac{1}{50} = \frac{2}{100},$$

and, for $x = \frac{3}{79}$,

$$\frac{x^2}{1+x^2} = \frac{9}{6250} = \frac{144}{100000}.$$

Substituting the equivalent series for the inverse tangents in equation (1), and multiplying by 4, we have

$$\pi = \frac{28}{10}\left[1 + \frac{2}{3}\,\frac{2}{100} + \frac{2\,4}{3\,5}\left(\frac{2}{100}\right)^2 \cdots\right]$$

$$+ \frac{30336}{100000}\left[1 + \frac{2}{3}\,\frac{144}{100000} + \frac{2\,4}{3\,5}\left(\frac{144}{100000}\right)^2 + \cdots\right].$$

For convenience of computation, we may write the series in

---

* This formula, suggested by Euler in 1779, (like the others which have been employed in the computation of π) is readily verified by means of the formula for the tangent of the sum of two arcs. Denoting the tangents of the arcs by m and n, the tangent of the sum is

$$\frac{m+n}{1-mn};$$

whence, putting $m = \frac{1}{7}$, and $n = \frac{3}{79}$, we obtain

$$\tan^{-1}\tfrac{1}{7} + \tan^{-1}\tfrac{3}{79} = \tan^{-1}\tfrac{2}{11}.$$

Hence

$$5\tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{3}{79} = 3\tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{2}{11}.$$

In like manner the second member may be reduced to $\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3}$, and finally to $\tan^{-1} 1$ or $\frac{1}{4}\pi$.

the following form, in which each of the letters α, β, γ, etc., denotes the value of the term preceding that in which it occurs:

$$\pi = \frac{14}{15}\left[3 + \frac{4}{100} + \frac{16}{10}\left(\frac{2}{100}\right)^2 + \frac{6}{7}\frac{2\alpha}{100} + \frac{8}{9}\frac{2\beta}{100} + \frac{10}{11}\frac{2\gamma}{100} + \cdots\right]$$

$$+ \frac{10112}{100000}\left[3 + \frac{288}{100000} + \frac{4}{5}\frac{144\alpha}{100000} + \frac{6}{7}\frac{144\beta}{100000} + \cdots\right].$$

This form indicates the method by which the value of each term is derived from that which precedes it. The numerical work for ten places of decimals is given below; multiplication by the factors $\frac{6}{7}$, $\frac{8}{9}$, etc., is effected by deducting $\frac{1}{7}$, $\frac{1}{9}$, etc., of the quantity to be multiplied.

$$\begin{array}{rr}
 & 3.04 \\
\alpha = & 0.00064 \\
.02\alpha = & 128 \\
\frac{1}{7}(.02\alpha) = & 182857 \\ \hline
\beta = & 1097143 \\
.02\beta = & 21943 \\
 & 2438 \\ \hline
\gamma = & 19505 \\
 & 390 \\
 & 35 \\ \hline
\delta = & 355 \\
 & 7 \\
 & 1 \\
\epsilon = & 6 \\ \hline
 & 3.04065117009 \\
 & .20271007801 \\ \hline
 & 2.83794109208
\end{array}$$

$$\begin{array}{rr}
 & 3. \\
\alpha = & 0.00288 \\
0.00144\,\alpha = & 41472 \\
 & 82944 \\ \hline
\beta = & 331776 \\
0.00144\,\beta = & 478 \\
 & 68 \\ \hline
\gamma = & 410 \\ \hline
 & 3.00288332186 \\
 & .10112 \\ \hline
 & 30028833219 \\
 & 300288332 \\
 & 30028833 \\
 & 6005767 \\ \hline
 & 0.30365156615 \\
 & 2.83794109208 \\ \hline
\pi = & 3.14159265365
\end{array}$$

Non-linear Differential Equations.

213. The differential equations employed in Arts. 207 and 209 are the results of eliminating transcendental and irrational functions from the immediate results of differentiation. It is because these equations, see for instance equation (4), Art. 209, constitute *linear* relations between the given function and its derivatives, that they give rise to simple relations between successive derivatives.

When the differential equation so found is not linear no such simple general relation can be found, but the results of successive differentiation may serve to determine the values of the derivatives, one after another. For example, given the function

$$y = e^{e^x}, \qquad \text{whence} \qquad \frac{dy}{dx} = e^x . e^{e^x}:$$

combining these equations, we have

$$\frac{dy}{dx} = ye^x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which gives the first derivative in an implicit form. Differentiating again,

$$\frac{d^2y}{dx^2} = e^x\left(y + \frac{dy}{dx}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We might obtain a differential equation free from the transcendental function by combining equations (1) and (2); but, in this case, direct differentiation gives a more convenient set of equations. Thus we have

$$\left.\begin{aligned} D^3y &= e^x(y + 2Dy + D^2y), \\ D^4y &= e^x(y + 3Dy + 3D^2y + D^3y), \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

where the coefficients are those of the Binomial Theorem, see equation (2), Art. 107.

When $x = 0$, the factor $e^x = 1$, and the initial value of the function is $y_0 = e$. Hence by equations (1), (2) and (3), e is a factor of each of the quantities y_0, $\left.\frac{dy}{dx}\right]_0$, $\left.\frac{d^2y}{dx^2}\right]_0$, etc., and the coefficients are readily found to be 1, 1, 2, 5, 15, 52, 203, etc. Hence the development

$$e^{e^x} = e[1 + x + x^2 + \tfrac{5}{6}x^3 + \tfrac{5}{8}x^4 + \tfrac{13}{30}x^5 + \cdots].$$

214. The process is similar when the original function is given in the implicit form. For example, in the equation

$$y^3 - 6xy - 8 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

y is a three-valued function of x, of which only one value however is real when $x = 0$, having then the value 2. Hence as x varies, starting from the value zero, one value of the function varies continuously, starting with the initial value 2. To develop this value in powers of x, we have, by differentiation of equation (1),

$$(y^2 - 2x)\frac{dy}{dx} - 2y = 0; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence, putting $x = 0$ and $y = 2$, we find $\left.\frac{dy}{dx}\right]_0 = 1$. Again differentiating,

$$2y\left(\frac{dy}{dx}\right)^2 - 4\frac{dy}{dx} + (y^2 - 2x)\frac{d^2y}{dx^2} = 0; \quad . \quad . \quad . \quad (3)$$

whence, substituting the values already found, $\left.\frac{d^2y}{dx^2}\right]_0 = 0$. In like manner we can find the values

$$\left.\frac{d^3y}{dx^3}\right]_0 = -\frac{1}{2}, \qquad \left.\frac{d^4y}{dx^4}\right]_0 = 1, \text{ etc.}$$

Hence the required expansion is

$$y = 2 + x - \tfrac{1}{12}x^3 + \tfrac{1}{24}x^4 + \cdots$$

Examples XXI.

1. Derive the expansion of $\cot^{-1}\dfrac{x}{a}$ from equation (5), Art. 104.

$$\cot^{-1}\frac{x}{a} = \frac{\pi}{2} - \frac{x}{a} + \frac{1}{3}\frac{x^3}{a^3} - \frac{1}{5}\frac{x^5}{a^5} + \cdots$$

2. If $y = \sin^{-1}x$, derive the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0,$$

and thence the expansion of the function.

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots$$

3. Expand $e^x \sin x$.

$$e^x \sin x = x + x^2 + \frac{2x^3}{3!} - \frac{4x^5}{5!} - \frac{8x^6}{6!} - \frac{8x^7}{7!} + \frac{16x^9}{9!} + \cdots$$

4. Expand $\log\,[x + \sqrt{(a^2 + x^2)}]$.

$$\log\,[x + \sqrt{(a^2 + x^2)}] = \log a + \frac{x}{a} - \frac{1}{2}\frac{x^3}{3a^3} + \frac{1.3}{2.4}\frac{x^5}{5a^5} - \cdots$$

5. By means of the series given in Art. 208 derive values of $\sin 5x$, $\sin 7x$, $\cos 4x$, $\cos 6x$.

$$\begin{aligned}
\sin 5x &= 5 \sin x - 20 \sin^3 x + 16 \sin^5 a;\\
\sin 7x &= 7 \sin x - 56 \sin^3 x + 112 \sin^5 x - 64 \sin^7 x;\\
\cos 4x &= 1 - 8 \sin^2 x + 8 \sin^4 x;\\
\cos 6x &= 1 - 18 \sin^2 x + 48 \sin^4 x - 32 \sin^6 x.
\end{aligned}$$

6. The function $(\cos^{-1}x)^2$ satisfies the same differential equation with $(\sin^{-1}x)^2$, see Ex. XI. 17 and Art. 209. Thence expand $(\cos^{-1}x)^2$.

Notice that, in accordance with the relation

$$(\cos^{-1}x)^2 = (\tfrac{1}{2}\pi - \sin^{-1}x)^2 = \frac{\pi^2}{4} - \pi \sin^{-1}x + (\sin^{-1}x)^2,$$

the terms of uneven degree give rise to the expansion of $\sin^{-1}x$. Compare Ex. 2 above.

7. Expand $[x + \sqrt{(a^2 + x^2)}]^m$.

$$\begin{aligned}
[x + \sqrt{(a^2 + x^2)}]^m &= a^m + ma^{m-1}x + \frac{m^2}{2!}a^{m-2}x^2 + \frac{m(m^2-1)}{3!}a^{m-3}x^3\\
&\quad + \frac{m^2(m^2-4)}{4!}a^{m-4}x^4 + \frac{m(m^2-1)(m^2-9)}{5!}a^{m-5}x^5 + \dots
\end{aligned}$$

8. Given that $y = a \cos \log x + b \sin \log x$ satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

(which is readily verified), derive the expansions by Taylor's Theorem of $\cos \log (1 + h)$ and of $\sin \log (1 + h)$.

$$\cos \log (1 + h) = 1 - \frac{h^2}{2!} + \frac{3h^3}{3!} - \frac{10h^4}{4!} + \frac{40h^5}{5!} - \dots$$

$$\sin \log (1 + h) = h - \frac{h^2}{2!} + \frac{h^3}{3!} - \frac{10h^5}{5!} + \frac{90h^6}{6!} - \dots$$

9. Expand $\exp(m\sin^{-1}x)$.

$$\exp(m\sin^{-1}x)=1+mx+\frac{m^2x^2}{2!}+\frac{m(m^2+1)}{3!}x^3+\frac{m^2(m^2+4)}{4!}x^4+\cdots$$

10. Given the differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0,$$

and

$$y_0=a,$$

to expand y in powers of x.

$$y=a\left[1-x+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+\frac{x^4}{(4!)^2}-\frac{x^5}{(5!)^2}+\cdots\right].$$

11. Expand to five terms the implicit function defined by the equation

$$e^y+xy=e.$$

$$y=1-\frac{x}{e}+\frac{1}{2!}\frac{x^2}{e^2}+\frac{1}{3!}\frac{x^3}{e^3}-\frac{10}{4!}\frac{x^4}{e^4}+\cdots$$

XXII.

Even and Uneven Functions.

215. Functions of x of which the value is not changed when x is changed to $-x$ are called *even functions*, because their developments in integral powers of x can contain only terms with even exponents. Functions of which the sign, but not the numerical value, is changed when x is changed to $-x$ are called *uneven functions*, because their developments can contain only terms with uneven exponents. In other words, even functions are those which have the property

$$f(-x) = f(x); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and uneven functions are those which have the property

$$f(-x) = -f(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Putting $y = f(x)$, equation (1) shows that the graph of an even function is symmetrical to the axis of y; for, if the point (a, b) is on the curve, the point $(-a, b)$ is also on the curve. See, for example, the graphs of $\cos x$ and $\sec x$, Figs. 12 and 14, pp. 63 and 65. Again, equation (2) shows that the graph of an uneven function is symmetrical to the origin as a centre; for if (a, b) is a point of the curve, $(-a, -b)$ is also a point of the curve. See the graphs of $\sin x$ and $\tan x$, Figs. 12 and 13. If an odd function is continuous through the value $x = 0$, the origin is a point of the curve. By differentiating equations (1) and (2), we see that the derivative of an even function is an uneven one, and that the derivative of an uneven function is an even one. This is also evident on differentiating the developments.

For example, $\sin x$ and $\tan x$ are uneven functions, and their derivatives, $\cos x$ and $\sec^2 x$, are even functions.

216. The development of a function $f(x)$ which belongs to neither of these classes may be separated into two series, one containing the even and the other the uneven powers of x. Denoting the sums, or generating functions, of these series by $\phi(x)$ and $\psi(x)$ respectively, we have

$$f(x) = \phi(x) + \psi(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which ϕ and ψ denote respectively an even and an uneven function. It follows that

$$f(-x) = \phi(x) - \psi(x), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and, combining equations (1) and (2),

$$\phi(x) = \tfrac{1}{2}[f(x) + f(-x)], \quad . \quad . \quad . \quad . \qquad (3)$$

$$\psi(x) = \tfrac{1}{2}[f(x) - f(-x)]. \quad . \quad . \quad . \quad . \qquad (4)$$

Thus from any function f admitting of development we can derive an even and an uneven function defined by equations (3) and (4).

Hyperbolic Functions.

217. The even and the uneven function thus derived from the exponential function e^x are called respectively *the hyperbolic cosine* and *the hyperbolic sine* of x, and are denoted by the symbols cosh and sinh. Thus, putting $f(x) = e^x$,

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}); \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and, from the development of e^x, we have

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots, \quad . \quad . \quad . \quad . \qquad (3)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \quad . \quad . \quad . \quad . \qquad (4)$$

The ratio $\dfrac{\sinh x}{\cosh x}$ is called *the hyperbolic tangent* of x, and is denoted by $\tanh x$; so also the reciprocals of these three functions are denoted by $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\coth x$.

218. The graphs of the functions $\cosh x$ and $\sinh x$ are given in Fig. 31. The dotted lines are the exponential curves $y = e^x$ and $y = e^{-x}$, and the ordinate of the curve $y = \cosh x$* is the arithmetical mean between the corresponding ordinates of these two curves.

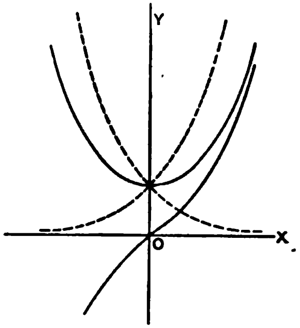

FIG. 31.

From equations (1) and (2), we find

$$\frac{d}{dx}\cosh x = \sinh x,$$

$$\frac{d}{dx}\sinh x = \cosh x,$$

and, since $\cosh 0 = 1$, the last equation shows that the graph of $\sinh x$ cuts the axis at the origin at an angle of 45°.

219. Relations exist between these functions bearing a remarkable analogy to those between the circular functions. For example,

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* The equation of *the catenary curve*, the form assumed by a perfectly flexible cord of uniform weight attached to two fixed points, is

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = c\cosh\frac{x}{c};$$

so that the graph of $\cosh x$ is the catenary in which $c = 1$.

$$\cosh^2 x - \sinh^2 x = 1,$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

These formulæ are readily verified by means of equations (1) and (2); they may also be derived from the corresponding trigonometric formulæ, see Art. 220 below.

Functions of Imaginary Quantities.

220. The product of a real quantity x by the imaginary factor $\sqrt{(-1)}$ is called a *pure imaginary* quantity. Denoting the imaginary factor by $i = \sqrt{(-1)}$, we have

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad \text{etc.}$$

The function $f(ix)$ of the pure imaginary variable ix may be defined as the result of substituting ix for x in the development of $f(x)$. For example, from the developments of $\cos x$ and $\sin x$, Art. 188, we obtain

$$\cos ix = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots,$$
$$\sin ix = i\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right];$$

hence, comparing with Art. 217, we find

$$\cos ix = \cosh x, \qquad \sin ix = i \sinh x. \quad . \quad . \quad (1)$$

Dividing, we find

$$\tan ix = i \tanh x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and taking reciprocals, we have

$$\sec ix = \operatorname{sech} x, \qquad \operatorname{cosec} ix = -i \operatorname{cosech} x,$$

and

$$\cot ix = -i \coth x.$$

By means of these equations, a trigonometric formula assumed to hold true for the pure imaginary ix is converted into a formula connecting the hyperbolic functions of x.

221. It is obvious that an even function of the pure imaginary ix must always, as in the preceding article, be a real quantity, and that an uneven function of ix must always be a pure imaginary. As a further illustration, the development of $\tan^{-1}x$, Ex. XIX, 4, gives

$$\tan^{-1}ix = i(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \cdots). \quad . \quad . \quad (1)$$

Hence, comparing with equation (3), Art. 198, we have

$$\tan^{-1}ix = \frac{i}{2} \log \frac{1 + x}{1 - x}. \quad . \quad . \quad . \quad . \quad (2)$$

Now, if in equation (2), Art. 220, we put $y = \tanh x$, so that $x = \tanh^{-1}y$, we have

$$\tan ix = iy; \quad \therefore \quad \tan^{-1}iy = ix = i \tanh^{-1}y.$$

Hence equation (2) above gives

$$\tanh^{-1}x = \tfrac{1}{2} \log \frac{1 + x}{1 - x}. \quad . \quad . \quad . \quad . \quad (3)$$

222. In general a given function f of the pure imaginary ix consists of a real and a pure imaginary part, which can be separated by developing the function f. Thus, from the exponential series, we find

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots,$$

which may be written

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$+ i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right].$$

Here we notice that the real part is the expansion of $\cos x$ and the coefficient of i is the expansion of $\sin x$; therefore

$$e^{ix} = \cos x + i \sin x. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Changing the sign of x,

$$e^{-ix} = \cos x - i \sin x. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Expressions of this kind, consisting of a real and a pure imaginary part, are called *complex imaginary* (in distinction from pure imaginary) quantities, or simply *complex quantities*.

Equations (1) and (2) give

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}),$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}),$$

in which, by the introduction of complex quantities, the real quantities $\cos x$ and $\sin x$ are expressed in *exponential forms*.

Complex Quantities.

223. Denoting any complex quantity by $x+iy$, where x and y are real, it is completely represented by the point whose rectangular coordinates are x and y, that is to say by the position of that point with reference to the origin and axes. Let ρ and θ be the polar coordinates of the point, so that

$$x=\rho\cos\theta,\quad y=\rho\sin\theta;$$

then the complex quantity may be written in the form

$$x+iy=\rho(\cos\theta+i\sin\theta).$$

Therefore by equation (1) of the preceding article we have

$$x+iy=\rho e^{i\theta}. \quad . \quad . \quad (1)$$

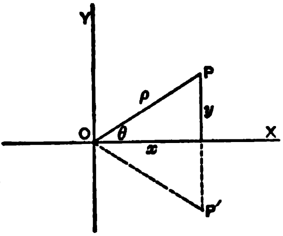

FIG. 32.

In this form, ρ is called the ***modulus***, and θ the ***argument*** of the complex quantity. Thus if P, Fig. 32, is the representative point (x, y), the modulus is the distance OP of the point from the origin, and the argument θ is the circular measure of the angle XOP which determines the direction of OP. We have also

$$\rho=\surd(x^2+y^2),\qquad \tan\theta=\frac{y}{x}.$$

When x and y are given, ρ is always taken as positive, and therefore θ must be so taken that its sine has the algebraic sign of y, and its cosine that of x. Thus θ has but one value between 0 and 2π, and this value may be called *the primary value* of the argument.

For a real negative quantity, the representing point is on the axis of x to the left of the origin; hence the primary value of the argument is π. For the pure imaginary i, it is $\frac{1}{2}\pi$; and for $-i$, it is $\frac{3}{2}\pi$.

Conjugate Complex Quantities.

224. The complex quantity $x - iy$ is called *the conjugate* of $x + iy$. It is represented by the point P' in Fig. 32 situated symmetrically to P with respect to the axis of x; and we have

$$x - iy = \rho\,(\cos\theta - i\sin\theta) = \rho e^{-i\theta}. \quad . \quad . \quad . \quad (1)$$

Thus conjugate complex quantities may be defined as having the same modulus and equal arguments with opposite signs. The sum of the conjugate quantities $x \pm iy$ is the real quantity $2x$, and their product is the positive quantity ρ^2. When the roots of an ordinary quadratic equation are imaginary, they are conjugates.

The complex quantity $\cos\theta + i\sin\theta$, of which the modulus is unity, is called a *complex unit*. A complex unit is therefore the exponential of a pure imaginary, the real coefficient of i in the exponent being the argument. Equations (1) and (2) of Art. 222 express *conjugate units* and show that they are reciprocals each of the other.

Functions of the Complex Quantity.

225. Equation (1) of Art. 223 gives

$$\log\,(x + iy) = \log\rho + i\theta,$$

in which $\log\rho$ is real because ρ is a positive quantity. Thus the logarithm of a complex quantity is a complex quantity of which the real part is the logarithm of the given modulus and the coefficient of i is the given argument. We have seen in Art. 223 that, for a given complex quantity, the argument θ has but one value between 0 and 2π; but, denoting this primary value by θ', θ admits of the multiple values $2k\pi + \theta'$, where k is any positive or negative integer. Thus,

$$\log(x + iy) = \tfrac{1}{2}\log(x^2 + y^2) + i(2k\pi + \theta');$$

whence it appears that the logarithm is a multiple-valued function, having the pure imaginary period $2i\pi$. The value obtained by putting $k = 0$ may be called *the primary value* of the function; for example, the primary value of the logarithm of a real negative quantity is the ordinary logarithm of its numerical value increased by $i\pi$.

226. From equation (1), Art. 223, we have also

$$(x + iy)^m = \rho^m e^{im\theta}.$$

That is to say, to raise a complex quantity to the mth power the modulus is raised to the mth power, and the argument is multiplied by m. The modulus is, for this reason, regarded as the absolute value of the complex quantity; hence any power of a complex unit (as defined in Art. 224) is a unit.

De Moivre's Theorem.

227. Substituting for the complex units in the identity

$$e^{im\theta} = (e^{i\theta})^m$$

their values in the form (1), Art. 222, we have

$$\cos m\theta + i\sin m\theta = (\cos\theta + i\sin\theta)^m. \quad . \quad . \quad (1)$$

The result is known as *De Moivre's Theorem.* It gives, by means of the Binomial Theorem, expressions for cos $m\theta$ and sin $m\theta$ in terms of cos θ and sin θ.

The number of terms in each of these expansions will be finite when m is a positive integer. Thus, putting $m = 3$ in equation (1), we obtain

$$\cos 3\theta + i \sin 3\theta = \cos^3\theta + 3 \cos^2\theta \,.\, i \sin \theta - 3 \cos \theta .\, \sin^2\theta - i \sin^3\theta.$$

Whence

$$\cos 3\theta = \cos^3\theta - 3 \cos \theta \sin^2\theta,$$

and

$$\sin 3\theta = 3 \cos^2\theta \sin \theta - \sin^3\theta.$$

De Moivre's Theorem also gives us the mth power of the complex variable $x + iy$, in the form $X + iY$, without expanding the binomial. For

$$(x + iy)^m = \rho^m(\cos \theta + i \sin \theta)^m = \rho^m(\cos m\theta + i \sin m\theta),$$

which gives X and Y in finite form, even when m is not an integer. For example,

$$(2 + i)^{\frac{3}{2}} = 5^{\frac{3}{4}} (\cos \tfrac{3}{2} \tan^{-1} \tfrac{1}{2} + i \sin \tfrac{3}{2} \tan^{-1} \tfrac{1}{2}).$$

228. When m is the reciprocal of an integer, say $m = \dfrac{1}{n}$, equation (1) becomes

$$\sqrt[n]{(\cos \theta + i \sin \theta)} = \cos \frac{\theta}{n} + i \sin \frac{\theta}{n}. \quad . \quad . \quad (2)$$

That is to say, the *n*th root of a complex unit is found by dividing the argument by *n*.

Now, if θ in equation (2) denotes the primary value of the argument of the given complex unit, we have seen that it also admits of any of the values included in the expression $\theta + 2k\pi$; hence the argument of the *n*th root admits of any one of the values included in the expression

$$\frac{\theta}{n} + \frac{2k\pi}{n}.$$

Giving to k the successive values $1, 2, 3, \ldots (n-1)$, we obtain $n - 1$ new values of the argument of the root, each of which is less than 2π. We have thus n distinct angles, namely,

$$\frac{\theta}{n}, \quad \frac{\theta + 2\pi}{n}, \quad \frac{\theta + 4\pi}{n}, \ldots \frac{\theta + 2(n-1)\pi}{n},$$

each of which is the primary value of the argument of a distinct value of $(\cos\theta + i\sin\theta)^{\frac{1}{n}}$.

229. These angles are derived by successive additions of the angle $2\pi/n$. If we continue the process we obtain only angles which differ by 2π or a multiple of 2π from those already written, so that they form other values of the arguments of which the primary values are those written above. Hence we have n, and only n, distinct values of the *n*th root of the given complex unit, namely,

$$\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}, \quad \cos\frac{\theta + 2\pi}{n} + i\sin\frac{\theta + 2\pi}{n}, \ldots$$
$$\cos\frac{\theta + 2(n-1)\pi}{n} + i\sin\frac{\theta + 2(n-1)\pi}{n}.$$

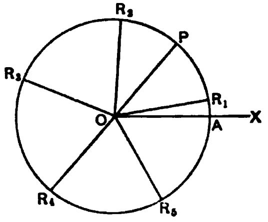

FIG. 33.

The geometrical representatives of these n values of the nth root form an equiangular set of radii of the unit circle, as in Fig. 33, where OP represents the given complex unit, and OR_1, OR_2, OR_3, OR_4 and OR_5, the five fifth roots of OP.

230. As a particular case, when $\theta = 0$, the given complex unit becomes the real unit $+1$, and the expressions for the roots becomes

$$1,\quad \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n},\ldots$$
$$\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n},$$

the last of which may be also written $\cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}$.

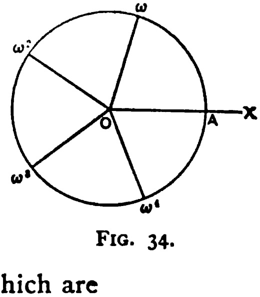

FIG. 34.

Thus $+1$ is the primary nth root of unity; if the first of the imaginary roots is denoted by ω, the remaining roots are $\omega^2, \omega^3, \ldots \omega^{n-1}$, all of which are imaginary, except when n is an even number, in which case the root $\omega^{\frac{n}{2}}$ has the value -1. It will be noticed that the roots ω and ω^{n-1} (or ω^{-1}) are conjugates, and so also the other imaginary roots occur in conjugate pairs.

Fig. 34 gives the geometrical representation of the fifth roots of unity, which are

$$+1,\quad \cos 72^\circ \pm i\sin 72^\circ,\quad \cos 144^\circ \pm i\sin 144^\circ.$$

The nth root of a complex quantity is an n-valued function of which the several values can be obtained by multiplying any one of them by the several nth roots of unity.*

Quadratic Factors of Certain Algebraic Expressions.

231. The equation of which the roots are the expressions found in Art. 230 is $x^n - 1 = 0$; hence we have

$$x^n - 1 = (x - 1)\left(x - \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)$$
$$\left(x - \cos\frac{4\pi}{n} - i\sin\frac{4\pi}{n}\right)\cdots\left(x - \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right). \quad (1)$$

The product of the factors corresponding to the first and last of the imaginary roots is the real quadratic factor

$$x^2 - 2x\cos\frac{2\pi}{n} + 1,$$

and combining, in like manner, the factors corresponding to each pair of conjugate roots, we have, when n is odd,

$$x^n - 1 = (x - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\cdots$$
$$\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right), \quad (2)$$

---

* In like manner, when $m = \frac{p}{q}$, p and q being integers, the function $(x+iy)^m$ is a q-valued function. But if m is incommensurable, the number of values obtained, as in Art. 229, by using different values of the argument is unlimited. In this case, we can only deal with the single-valued function corresponding to the primary value of the argument of $x+iy$.

and when n is even,

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\cdots$$

$$\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)(x+1). \quad (3)$$

232. The roots of the equation

$$x^{2n} - 2x^n\cos\theta + 1 = 0 \quad . \quad . \quad . \quad . \quad (1)$$

are the roots of the two equations

$$x^n = \cos\theta \pm i\sin\theta,$$

found by solving it as a quadratic for x^n. They are, therefore, the n quantities found in Art. 229, together with their conjugates, which are the results of changing the sign of θ. The first member of equation (1) is thus separated into $2n$ linear factors. Combining, as in Art. 231, the factors corresponding to the conjugate pairs of roots, we obtain

$$x^{2n} - 2x^n\cos\theta + 1 = \left(x^2 - 2x\cos\frac{\theta}{n} + 1\right)\left(x^2 - 2x\cos\frac{2\pi+\theta}{n} + 1\right)\cdots$$

$$\left(x^2 - 2x\cos\frac{2(n-1)\pi+\theta}{n} + 1\right) \quad . \quad . \quad (2)$$

Examples XXII.

1. Show that $\log[x + \sqrt{(1+x^2)}]$ is an uneven function, and that $\log\ x + \sqrt{(a^2+x^2)}]$ is an uneven function increased by a constant.

2. Prove that each of the following expressions denotes an even function of x:

$$x\cot x \qquad \text{and} \qquad \frac{x}{2} + \frac{x}{e^x - 1},$$

3. Prove that the following denote uneven functions of x:

$$\log \frac{1+x}{1-x}, \qquad \log \tan \left(\frac{1}{4}\pi + x\right).$$

4. Show that if ϕ denotes a one-valued function, $\phi(x^2)$ is an even function of x. Compare Ex. I. 22.

5. Show that an uneven function of an uneven function is an uneven function, and that the product of two uneven functions is an even one.

6. Prove the relations:

$$\sinh 2x = 2 \sinh x \cosh x, \qquad \text{and} \qquad \cosh 2x = \cosh^2 x + \sinh^2 x.$$

7. Prove the formulæ:

$$d \tanh x = \operatorname{sech}^2 x \, dx, \qquad d \operatorname{sech} x = - \tanh x \operatorname{sech} x \, dx$$

$$d \coth x = - \operatorname{cosech}^2 x \, dx \qquad d \operatorname{cosech} = - \coth x \operatorname{cosech} x \, dx$$

8. Find the hyperbolic sine, cosine and tangent of the pure imaginary ix.

$$\sinh (ix) = i \sin x,$$
$$\cosh (ix) = \cos x,$$
$$\tanh (ix) = i \tan x.$$

9. Express the sine, cosine and tangent of the complex quantity $x + iy$ in the form $X + iY$.

$$\sin (x + iy) = \sin x \cosh y + i \cos x \sinh y,$$
$$\cos (x + iy) = \cos x \cosh y - i \sin x \sinh y,$$
$$\tan (x + iy) = \frac{\tan x \operatorname{sech}^2 y}{1 + \tan^2 x \tanh^2 y} + i \frac{\sec^2 x \tanh y}{1 + \tan^2 x \tanh^2 y}.$$

10. Derive the value of $\tanh^{-1} x$, equation (3), Art. 221, directly from the exponential expression for $\tanh y$.

11. If $f(x) = \log \frac{1+x}{1-x}$, prove directly that

$$f(x) - f(y) = f\left(\frac{x-y}{1-xy}\right);$$

also, using the notation of hyperbolic functions, prove the equation by means of the relations given in Art. 219.

12. Prove that

$$[\sqrt{(a^2 - x^2)} + ix]^m = a^m\left[\cos m \sin^{-1}\frac{x}{a} + i \sin m \sin^{-1}\frac{x}{a}\right].$$

Compare the results of putting ix for x in the developments given in Ex. XXI. 7 and in Art. 208.

13. Prove that

$$\sin^{-1}ix = i \log [x + \sqrt{(1 + x^2)}] = i \sinh^{-1}x;$$

also that

$$\sinh^{-1}\frac{x}{a} = \log \frac{x + \sqrt{(a^2 + x^2)}}{a}.$$

14. Deduce the derivatives of the functions $\sinh^{-1}x$ and $\cosh^{-1}x$ from formulæ given in Arts. 218 and 219.

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{(1 + x^2)}}, \quad \frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{(x^2 - 1)}}.$$

15. Prove that the inverse hyperbolic cosine is the two-valued function $\pm \cosh^{-1}x$, where

$$\cosh^{-1}x = \log [x + \sqrt{(x^2 - 1)}].$$

16. Derive the development of $\sinh^{-1}x$ from that of its derivative as in Art. 185.

$$\sinh^{-1}x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots$$

Compare Exs. XXI. 2 and 4.

17. Show that $\sinh^{-1}\frac{1}{x} = \operatorname{cosech}^{-1}x = \log \frac{1 + \sqrt{(1 + x^2)}}{x}$.

18. Develop the function $\sinh^{-1}\frac{1}{x} + \log x$ in ascending powers of x, and thence derive a series for $\sinh^{-1}x$ which is convergent when $x > 1$.

$$\sinh^{-1}x = \log 2x + \frac{1}{2}\frac{1}{2x^2} - \frac{1.3}{2.4}\frac{1}{4x^4} + \frac{1.3.5}{2.4.6}\frac{1}{6x^6} - \ldots$$

19. Develop $\cosh^{-1}x$ in similar form.

$$\cosh^{-1}x = \log 2x - \frac{1}{2}\frac{1}{2x^2} - \frac{1.3}{2.4}\frac{1}{4x^4} - \frac{1.3.5}{2.4.6}\frac{1}{6x^6} - \ldots$$

20. Show, by direct multiplication of the complex quantities $a + ib$ and $c + id$, that the modulus of the product is the product of the moduli, and that the argument of the product is the sum of the arguments.

21. Show, by means of the multiple values of log i, that the symbol $\sqrt[i]{i}$ represents the real quantity $\sqrt{e^{\pi}}$, or the product of this by an integral power of $e^{2\pi}$.

22. Find expressions for sin $5x$ and cos $5x$ by means of De Moivre's theorem.

$$\sin 5x = 5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x.$$
$$\cos 5x = \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x.$$

23. Find expressions for sin mx and cos mx by means of De Moivre's theorem.

$$\cos mx = \cos^m x - \frac{m(m-1)}{1 \cdot 2} \cos^{m-2} x \sin^2 x + \cdots;$$
$$\sin mx = m \cos^{m-1} x \sin x - \frac{m(m-1)(m-2)}{1.2.3} \cos^{m-3} x \sin^3 x + \cdots$$

24. Find the sixth roots of unity.

The roots are ± 1, $\pm \frac{1}{2}(1 \pm i\sqrt{3})$.

25. Find the fourth roots of -1.

The roots are $\pm \frac{1}{2}\sqrt{2}(1 \pm i)$.

26. Find the cube roots of i.

The roots are $-i$, $\pm \frac{1}{2}\sqrt{3} + \frac{1}{2} i$.

27. Resolve $x^n + 1$ into factors.

When n is even,

$$x^n + 1 = \left(x^2 - 2x \cos \frac{\pi}{n} + 1\right)\left(x^2 - 2x \cos \frac{3\pi}{n} + 1\right) \cdots$$
$$\left(x^2 - 2x \cos \frac{n-3}{n}\pi + 1\right)\left(x^2 - 2x \cos \frac{n-1}{n}\pi + 1\right);$$

when n is odd,

$$x^n + 1 = \left(x^2 - 2x \cos \frac{\pi}{n} + 1\right) \cdots \left(x^2 - 2x \cos \frac{n-2}{n}\pi + 1\right)(x + 1).$$

XXIII.

The Sine and Cosine as Continued Products.

233. Putting $x = 1$ in equation (4), Art. 232, we obtain

$$2(1 - \cos\theta) = 2^n\left(1 - \cos\frac{\theta}{n}\right)\left(1 - \cos\frac{2\pi + \theta}{n}\right)$$
$$\left(1 - \cos\frac{4\pi - \theta}{n}\right)\cdots\left(1 - \cos\frac{2(n-1)\pi + \theta}{n}\right)$$

Using the formula $1 - \cos x = 2\sin^2\frac{1}{2}x$, and putting ϕ for $\frac{1}{2}\theta$,

$$4\sin^2\phi = 4^n\sin^2\frac{\phi}{n}\sin\left(\frac{\pi}{n} + \frac{\phi}{n}\right)\sin^2\left(\frac{2\pi}{n} + \frac{\phi}{n}\right)\cdots$$
$$\sin\left[\frac{(n-1)\pi}{n} + \frac{\phi}{n}\right].$$

Supposing $\theta < 2\pi$, and therefore $\phi < \pi$, each of the sines in this equation is a positive quantity; hence, taking the square root of each member, we have

$$\sin\phi = 2^{n-1}\sin\frac{\phi}{n}\sin\left(\frac{\pi}{n} + \frac{\phi}{n}\right)\cdots\sin\left[\frac{(n-1)\pi}{n} + \frac{\phi}{n}\right]. \quad (1)$$

If we add π to ϕ, the first member changes sign; but, in the second member, the first factor assumes the present value of the second, the second assumes that of the third, and so on, while the final factor becomes $\sin\left(\pi + \frac{1}{n}\phi\right)$, which has the present value of the first factor with its sign changed; therefore the second member also changes sign. Hence equation (1) applies, without change of sign, when $\phi > \pi$.

234. Since the sine of an angle is equal to the sine of its supplement, the last factor in equation (1) may be written $\sin\left(\frac{\pi}{n}-\frac{\phi}{n}\right)$. It may therefore be combined with the second factor by using the formula

$$\sin(x+y)\sin(x-y)=\sin^2 x-\sin^2 y.$$

In a similar manner the third factor and the last but one may be combined, and so on; therefore, if n is an odd number,

$$\sin\phi = 2^{n-1}\sin\frac{\phi}{n}\left(\sin^2\frac{\pi}{n}-\sin^2\frac{\phi}{n}\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\frac{\phi}{n}\right)\cdots \left(\sin^2\frac{n-1}{2}\frac{\pi}{n}-\sin^2\frac{\phi}{n}\right).^{*} \qquad (2)$$

Dividing this equation by $\sin\phi$, and then making $\phi = 0$, we have

$$1=\frac{2^{n-1}}{n}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\cdots\sin^2\frac{(n-1)\pi}{2n}. \quad . \quad . \qquad (3)$$

Again, dividing equation (2) by this last equation, we have

$$\sin\phi=n\sin\frac{\phi}{n}\left[1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{\pi}{n}}\right]\left[1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{2\pi}{n}}\right]\cdots\left[1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{(n-1)\pi}{2n}}\right].$$

---

* If n were even, the last quadratic factor would take the form

$$\sin^2\frac{n-2}{2}\frac{\pi}{n}-\sin^2\frac{\phi}{n},$$

and there would be the single factor $\cos\frac{\phi}{n}$ outstanding. The corresponding factor in equation (3) would be unity, and so also in equation (4), when n is made infinite.

Finally, if in the above equation ϕ remains fixed while n increases without limit, we have, on evaluation,

$$\sin\phi = \phi\left[1 - \frac{\phi^2}{\pi^2}\right]\left[1 - \frac{\phi^2}{2^2\pi^2}\right]\left[1 - \frac{\phi^2}{3^2\pi^2}\right]\cdots, \quad . \quad (4)$$

the number of factors being unlimited.

235. It will be noticed that this expression for $\sin\phi$ consists of a set of factors one of which vanishes when ϕ takes any one of the values which make $\sin\phi = 0$, namely,

$$0, \quad \pm\pi, \quad \pm 2\pi, \quad \text{etc.}$$

Again, arranging the expression in the form

$$\sin\phi = \cdots\frac{2\pi - \phi}{2\pi}\,\frac{\pi - \phi}{\pi}\,\frac{\phi}{1}\,\frac{\pi + \phi}{\pi}\,\frac{2\pi + \phi}{2\pi}\cdots \quad . \quad (5)$$

(the series of factors extending to infinity in both directions*), we see that changing ϕ to $\pi + \phi$, and moving each numerator one place to the right, the expression is reproduced with its sign changed. A second addition of π to the independent variable restores the original value of the product, thus proving the periodic character of the function, that is to say, the property $\sin\phi = \sin(2\pi + \phi)$.

---

* Although the factors both on the right and on the left of ϕ in equation (5) approach unity as a limit, the product of those on the right (as will be shown in the Integral Calculus) is infinite, while that of those on the left has zero for its limit. In the deduction of the equation an equal number of factors on each side is taken, and that number then becomes infinite. The inclusion of a finite number of factors in excess on one side would not affect the value of the product; but an infinite number would. An infinite number of these factors beginning at a point infinitely distant has in fact a finite product. Thus it will be shown that if n factors on the left and rn factors on the right were taken, and then n made infinite (r having a fixed value greater than unity), the product of the extra factors on the right would be r^m, where $m = \frac{\phi}{\pi}$.

236. A similar expression for $\cos\phi$ may be derived from equation (4) of Art. 234 by means of the formula

$$\cos\phi = \frac{\sin 2\phi}{2\sin\phi},$$

whence

$$\cos\phi = \frac{2\phi\left(1 - \frac{4\phi^2}{\pi^2}\right)\left(1 - \frac{4\phi^2}{4\pi^2}\right)\left(1 - \frac{4\phi^2}{9\pi^2}\right)\left(1 - \frac{4\phi^2}{16\pi^2}\right)\cdots}{2\phi\left(1 - \frac{\phi^2}{\pi^2}\right)\left(1 - \frac{\phi^2}{4\pi^2}\right)\cdots};$$

and, removing common factors, we have

$$\cos\phi = \left[1 - \frac{4\phi^2}{\pi^2}\right]\left[1 - \frac{4\phi^2}{3^2\pi^2}\right]\left[1 - \frac{4\phi^2}{5^2\pi^2}\right]\cdots \quad . \quad (1)$$

This equation may also be written in the form

$$\cos\phi = \cdots\frac{5\pi - 2\phi}{5\pi}\,\frac{3\pi - 2\phi}{3\pi}\,\frac{\pi - 2\phi}{\pi}\,\frac{\pi + 2\phi}{\pi}\,\frac{3\pi + 2\phi}{3\pi}\,\frac{5\pi + 2\phi}{5\pi}\cdots, \quad (2)$$

which exhibits the periodicity of the function, and also the values for which it vanishes.

237. If, in equation (4), Art. 234, we put $\phi = \frac{1}{2}\pi$, we obtain

$$\frac{2}{\pi} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\cdots$$

$$= \frac{1}{2}\,\frac{3}{2}\,\frac{3}{4}\,\frac{5}{4}\,\frac{5}{6}\,\frac{7}{6}\cdots;$$

whence

$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\cdots,$$

which is Wallis's expression for $\frac{1}{2}\pi$.

238. The coefficient of ϕ^3 in the ordinary development of sin ϕ is $-\frac{1}{6}$; if we equate this to the coefficient of ϕ^3 in the expansion of the continued product in equation (4), we obtain

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Again, dividing this equation by 4, we have

$$\frac{\pi^2}{24} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and, subtracting from the preceding result,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

239. By taking logarithms, a continued product is converted into an infinite series. Thus, equation (4), Art. 234, gives

$$\log \sin \phi = \log \phi + \log \left(1 - \frac{\phi^2}{\pi^2}\right) + \log\left(1 - \frac{\phi^2}{2^2\pi^2}\right) + \cdots; \qquad (1)$$

whence, expanding the logarithms by means of the development of log $(1 - x)$, Art. 185, and collecting the terms, we have

$$\begin{aligned} \log \frac{\phi}{\sin \phi} = \frac{\phi^2}{\pi^2}&\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) \\ &+ \frac{1}{2}\frac{\phi^4}{\pi^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right) \\ &+ \frac{1}{3}\frac{\phi^6}{\pi^6}\left(1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots\right) \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

The numerical series in this result are all convergent; denoting their values by $S_2, S_4, S_6 \ldots$, the equation may be written

$$\log \phi - \log \sin \phi = \frac{S_2}{\pi^2}\phi^2 + \frac{1}{2}\frac{S_4}{\pi^4}\phi^4 + \frac{1}{3}\frac{S_6}{\pi^6}\phi^6 + \cdots {}^*. \quad (2)$$

This series is convergent for all values of ϕ between π and $-\pi$.

Bernoulli's Numbers.

240. A series of numbers which occur in the expansion of certain functions was introduced by James Bernoulli in 1687, and has been the subject of much subsequent investigation. *Bernoulli's numbers* are the values of the coefficients of $\dfrac{x^n}{n!}$ in the development of the function $\frac{1}{2}x\coth\frac{1}{2}x$; that is to say, putting

$$y = \tfrac{1}{2}x\frac{e^x + 1}{e^x - 1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

they are the values which the successive derivatives of y take when $x = 0$.

These values may be found by means of the differential equation satisfied by the function, as in Art. 208. From equation (1), we have

$$2(ye^x - y) = x + xe^x,$$

---

[*] It will be shown in Art. 244 that $S_2 = \frac{1}{6}\pi^2$ (see also Art. 238) and that $S_4 = \frac{1}{90}\pi^4$; hence this equation becomes

$$\log \phi - \log \sin \phi = \tfrac{1}{6}\phi^2 + \tfrac{1}{180}\phi^4 + \cdots$$

Values of this function, multiplied by the modulus of common logarithms, are often given in Trigonometric Tables, to facilitate finding the logarithmic sines of small angles.

whence, differentiating,

$$2\left[e^x\frac{dy}{dx}+e^xy-\frac{dy}{dx}\right]=1+e^x+xe^x, \quad . \quad . \quad . \quad (2)$$

and in general by Leibnitz' Theorem, Art. 105,

$$2\left[e^x\frac{d^my}{dx^m}+me^x\frac{d^{m-1}y}{dx^{m-1}}+\cdots+e^xy-\frac{d^my}{dx^m}\right]=me^x+xe^x. \quad (3)$$

Putting $x=0$, equation (2) gives $y_0=1$, and equation (3) becomes

$$mD^{m-1}y+\frac{m(m-1)}{2}D^{m-2}y+\cdots+mDy+1=\frac{m}{2}, \quad . \quad (4)$$

in which D^ry is put for $\left.\frac{d^ry}{dx^r}\right]_0$.

241. It is readily shown that y is an even function, hence Dy and all the odd-numbered derivatives vanish. It follows that two relations can be found connecting any given even derivative with lower derivatives; one by means of an even value of m and the other by means of an odd value of m. Thus, if we put $m=2n+1$, we have

$$(2n+1)D^{2n}y+\frac{(2n+1)2n(2n-1)}{3!}D^{2n-2}y+\cdots$$
$$+\frac{(2n+1)2n}{2}D^2y=\frac{2n-1}{2}; \quad . \quad . \quad (5)$$

and, if we put $m=2n+2$, we have

$$\frac{(2n+2)(2n+1)}{2}D^{2n}y+\frac{(2n+2)(2n+1)2n(2n-1)}{4!}D^{2n-2}y+\cdots$$
$$+\frac{(2n+2)(2n+1)}{2}D^2y=n. \quad . \quad . \quad (6)$$

242. The numerical value of $D^{2n}y$ is called the nth *Bernoullian number*. Since it is found that the values alternate in sign while D^2 is positive, the notation adopted is

$$B_n = (-1)^{n-1}D^{2n}y. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

With this notation, equation (5) becomes

$$B_n - n\frac{2n-1}{3}B_{n-1} + n\frac{(2n-1)(2n-2)(2n-3)}{3\cdot 4\cdot 5}B_{n-2} - \cdots$$
$$- (-1)^n nB_1 = -(-1)^n\frac{2n-1}{2(2n+1)}, \qquad (8)$$

and equation (6) becomes

$$B_n - \frac{2n(2n-1)}{3\cdot 4}B_{n-1} + \frac{2n(2n-1)(2n-2)(2n-3)}{3\cdot 4\cdot 5\cdot 6}B_{n-2} - \cdots$$
$$+ (-1)^n\frac{2n(2n-1)}{3\cdot 4}B_2 - (-1)^n B_1 = -(-1)^n\frac{n}{(n+1)(2n+1)}. \quad (9)$$

Either equation gives, when $n = 1$, $B_1 = \frac{1}{6}$, and when $n = 2$, $B_2 = \frac{1}{30}$. Substituting the value of B_1, equation (9)* becomes

$$B_n = \frac{2n(2n-1)}{3\cdot 4}B_{n-1} - \frac{2n(2n-1)(2n-2)(2n-3)}{3\cdot 4\cdot 5\cdot 6}B_{n-2} + \cdots$$
$$-(-1)^n\frac{2n(2n-1)}{3\cdot 4}B_2 + (-1)^n\frac{(n-1)(2n-1)}{6(n+1)(2n+1)}. \quad (10)$$

Giving to n the successive values 3, 4, etc., we find

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66},$$
$$B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \quad B_8 = \frac{3617}{510}.\dagger$$

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* This equation is the more convenient on account of the recurrence of like coefficients.

† The values of the Bernoullian numbers up to B_{31} have been determined, and are given in *Crelle's Journal*, vol. xx, p. 11.

The development of the function in equation (1) is, by equation (7),

$$\tfrac{1}{2}x\,\frac{e^x+1}{e^x-1} = 1 + B_1\frac{x^2}{2!} - B_2\frac{x^4}{4!} + B_3\frac{x^6}{6!} - B_4\frac{x^8}{8!} + \cdots$$

$$= 1 + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots$$

The Development of ϕ cot ϕ.

243. Putting ix in place of x in the even function just developed, we have by Art. 222

$$\tfrac{1}{2}ix\,\frac{e^{ix}+1}{e^{ix}-1} = \tfrac{1}{2}ix\,\frac{e^{\frac{1}{2}ix}+e^{-\frac{1}{2}ix}}{e^{\frac{1}{2}ix}-e^{-\frac{1}{2}ix}} = \tfrac{1}{2}x \cot \tfrac{1}{2}x.$$

Hence the development above gives

$$\tfrac{1}{2}x \cot \tfrac{1}{2}x = 1 - B_1\frac{x^2}{2!} - B_2\frac{x^4}{4!} - B_3\frac{x^6}{6!} - \cdots,$$

or putting ϕ for $\tfrac{1}{2}x$,

$$\phi \cot \phi = 1 - B_1\frac{2^2\phi^2}{2!} - B_2\frac{2^4\phi^4}{4!} - B_3\frac{2^6\phi^6}{6!} - \cdots \quad . \quad (1)$$

Another form of the development of ϕ cot ϕ results from taking the derivative of equation (2), Art. 239, which is

$$\frac{1}{\phi} - \cot\phi = \frac{2S_2}{\pi^2}\phi + \frac{2S_4}{\pi^4}\phi^3 + \frac{2S_6}{\pi^6}\phi^5 + \cdots;$$

whence

$$\phi\cot\phi = 1 - \frac{2S_2}{\pi^2}\phi^2 - \frac{2S_4}{\pi^4}\phi^4 - \frac{2S_6}{\pi^6}\phi^6 - \cdots \quad , \quad (2)$$

244. Comparing the coefficients in these two expressions for ϕ cot ϕ, we have, for all values of n,

$$B_n\frac{2^{2n}}{(2n)!} = \frac{2}{\pi^{2n}}S_{2n} = \frac{2}{\pi^{2n}}\left[1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots\right]; \quad . \quad (1)$$

whence

$$S_{2n} = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = \frac{(2\pi)^{2n} B_n}{2(2n)!},$$

which expresses the sum of any even powers of the reciprocals of the natural numbers in terms of Bernoulli's numbers. For example, when $n = 1$, we have the value of S_2 already given in Art. 238, and when $n = 2$, we find

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Equation (1) may also be written in the form

$$B_n = \frac{2(2n)!}{(2\pi)^{2n}} 1 + \frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \cdots \Big],^*$$

which shows that the Bernoullian numbers are all positive, and also that they increase rapidly with n; for B_n approaches (but always exceeds) the quantity $\frac{2(2n)!}{(2\pi)^{2n}}$.

Examples XXIII.

1. Show, by means of equation (3), Art. 234, and the result of putting $2n$ for n in that equation, that

$$1 = 2^{n-1} \sin^2 \frac{\pi}{2n} \sin^2 \frac{3\pi}{2n} \sin^2 \frac{5\pi}{2n} \cdots \sin^2 \frac{(n-1)\pi}{2n} \text{ or } \sin^2 \frac{(n-2)\pi}{2n},$$

where n is an integer. (Angles in the first quadrant only included: thus, for $n = 9$ and $n = 10$,

$$1 = 2^8 \sin^2 10^\circ \sin^2 30^\circ \sin^2 50^\circ \sin^2 70,$$

$$1 = 2^9 \sin^2 9^\circ \sin^2 27^\circ \sin^2 45^\circ \sin^2 63^\circ \sin^2 81^\circ.)$$

---

* The logarithms, to ten decimal places, of Bernoulli's numbers up to B_{250} have been calculated from this formula by Dr. Glaisher, and are published together with the first nine figures of their values in the *Cambridge Phil. Trans.*, vol. xii, p. 386 and p. 390.

The full value of B_{250} would contain 736 digits before the decimal point.

2. Putting $\phi=\frac{1}{4}\pi$ in equation (4), Art. 234, prove that the product of the even-numbered *pairs* of factors in Wallis's expression, Art. 237, is the value of $\frac{\pi}{2\sqrt{2}}$, and thence that the product of the odd-numbered pairs is $\sqrt{2}$.

3. Derive the following continued products:

$$\frac{\pi}{3}=\frac{6}{5}\,\frac{6}{7}\,\frac{12}{11}\,\frac{12}{13}\,\frac{18}{17}\,\frac{18}{19}\cdots,$$

$$\frac{2\pi}{3\sqrt{3}}=\frac{3}{2}\,\frac{3}{4}\,\frac{6}{5}\,\frac{6}{7}\,\frac{9}{8}\,\frac{9}{10}\cdots$$

4. Derive equation (3), Art. 238, from the continued product for $\cos\phi$.

5. Express the hyperbolic sine in the form of a continued product by putting ix for ϕ in equation (4), Art. 234.

$$\sinh x = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)\cdots$$

6. Express the hyperbolic cosine in the form of a continued product:

$$\cosh x = \left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{3^2\pi^2}\right)\left(1+\frac{4x^2}{5^2\pi^2}\right)\cdots$$

7. Show that $\frac{x}{e^x-1}=\frac{x}{2}\,\frac{e^x+1}{e^x-1}-\frac{1}{2}x$; whence, by Art. 242,

$$\frac{x}{e^x-1}=1-\frac{x}{2}+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\cdots$$

$$=1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}+\frac{x^6}{30240}-\cdots$$

8. Separate $\dfrac{2x}{e^{2x}-1}$ into partial fractions, and thence by means of the result in Ex. 7 find the development of $\dfrac{1}{e^x+1}$.

$$\frac{1}{e^x+1}=\frac{1}{2}-(2^2-1)B_1\frac{x}{2!}+(2^4-1)B_2\frac{x^3}{4!}-(2^6-1)B_3\frac{x^5}{6!}+\cdots$$

9. Obtain a development by adding those of $\dfrac{1}{e^x-1}$ and $\dfrac{1}{e^x+1}$.

$$\frac{1}{e^x-e^{-x}}=\frac{1}{2x}-(2-1)B_1\frac{x}{2!}+(2^3-1)B_2\frac{x^3}{4!}-(2^5-1)B_3\frac{x^5}{6!}+\cdots$$

10. Develop, by means of Ex. 8, $\dfrac{e^x-1}{e^x+1}=\dfrac{1}{1+e^{-x}}-\dfrac{1}{e^x+1}$.

$$\frac{e^x-1}{e^x+1}=\frac{2(2^2-1)B_1}{2!}x-\frac{2(2^4-1)B_2}{4!}x^3+\frac{2(2^6-1)B_3}{6!}x^5-\cdots,$$

or, employing the values of B_1, B_2, etc., given in Art. 242,

$$\frac{e^x-1}{e^x+1}=\frac{x}{2}-\frac{x^3}{24}+\frac{x^5}{240}-\frac{17x^7}{40320}+\frac{31x^9}{725760}-\cdots$$

11. Derive in like manner the development of $\dfrac{e^x+1}{e^x-1}$ from Ex. 7.

$$\frac{e^x+1}{e^x-1}=\frac{2}{x}+2B_1\frac{x}{2!}-2B_2\frac{x^3}{4!}+2B_3\frac{x^5}{6!}\cdots\cdots$$

12. Show that Exs. 9, 10 and 11 give the following developments of hyperbolic functions:

$$\begin{aligned}\operatorname{cosech} x&=\frac{1}{x}-B_1x+\frac{2^3-1}{2}B_2\frac{x^3}{3!}-\frac{2^5-1}{3}B_3\frac{x^5}{5!}+\cdots\\&=\frac{1}{x}-\frac{x}{6}+\frac{7x^3}{360}-\frac{31x^5}{15120}+\cdots\end{aligned}$$

$$\begin{aligned}\tanh x&=\frac{2^2(2^2-1)}{2!}B_1x-\frac{2^4(2^4-1)}{4!}B_2x^3+\frac{2^6(2^6-1)}{6!}B_3x^5-\cdots\\&=x-\frac{1}{3}x^3+\frac{2}{15}x^5-\frac{17}{315}x^7+\cdots\end{aligned}$$

$$\coth x = \frac{1}{x} + \frac{2^2 B_1}{2!}x - \frac{2^4 B_2}{4!}x^3 + \frac{2^6 B_3}{6!}x^5 - \dots$$

$$= \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \dots$$

13. Denoting the sum to infinity of the nth powers of the reciprocals of the odd numbers by S'_n, show (see Arts. 238 and 244) that

$$S'_{2n} = \frac{2^{2n}-1}{2^{2n}} S_{2n} = \frac{(2^{2n}-1)\pi^{2n} B_n}{2(2n)!}.$$

14. Derive an expansion from equation (1), Art. 236,

$$\log \sec \phi = \frac{2^2 S'_2}{\pi^2}\phi^2 + \frac{1}{2}\frac{2^4 S'_4}{\pi^4}\phi^4 + \frac{1}{3}\frac{2^6 S'_6}{\pi^6}\phi^6 + \cdots$$

$$= \frac{2(2^2-1)B_1}{2!}\phi^2 + \frac{1}{2}\frac{2^3(2^4-1)B_2}{4!}\phi^4 + \frac{1}{3}\frac{2^5(2^6-1)B_3}{6!}\phi^6 + \cdots$$

15. Express the development of log sin ϕ, Art. 239, in terms of Bernoulli's numbers, and that of log tan ϕ by adding log sec ϕ.

$$\log \sin \phi = \log \phi - \frac{2B_1}{2!}\phi^2 - \frac{1}{2}\frac{2^3 B_2}{4!}\phi^4 - \frac{1}{3}\frac{2^5 B_3}{6!}\phi^6 - \dots,$$

$$\log \tan \phi = \log \phi + \frac{2^2(2-1)B_1}{2!}\phi^2 + \frac{2^4(2^3-1)B_2}{2.4!}\phi^4 + \frac{2^6(2^5-1)B_3}{3.6!}\phi^6 + \cdots$$

16. Show, by means of the continued product in Ex. 5, that the expansion of $\log \dfrac{\sinh \phi}{\phi}$ consists of the terms of that of $\log \dfrac{\phi}{\sin \phi}$ (Art. 239) with alternate signs changed; and thence, in the notation of Bernoulli's numbers, that

$$\log \frac{\sinh \phi}{\sin \phi} = \frac{2^2 B_1}{2!}\phi^2 + \frac{1}{3}\frac{2^6 B_3}{6!}\phi^6 + \frac{1}{5}\frac{2^{10} B_5}{10!}\phi^{10} + \cdots$$

17. By means of the identity

$$\tan x = \cot x - 2 \cot 2x$$

derive the development of tan x from equation (1), Art. 243,

$$\tan x = \frac{2^2(2^2-1)B_1}{2!}x + \frac{2^4(2^4-1)B_2}{4!}x^3 + \frac{2^6(2^6-1)B_3}{6!}x^5 + \cdots$$

This development may also be obtained from that of Ex. 14 by taking derivatives.

18. By means of the identity

$$\operatorname{cosec} x = \cot \tfrac{1}{2}x - \cot x$$

derive the development of cosec x,

$$\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{1}{2}\,\frac{2^3-1}{3!}\,B_2x^3 + \frac{1}{3}\,\frac{2^5-1}{5!}\,B_3x^5 + \cdots$$

19. Derive the expansion of $\sec^2 x$ and that of $\operatorname{cosec}^2 x$ by taking derivatives.

$$\sec^2 x = 1 + \frac{3.2^4(2^4-1)B_2}{4!}x^2 + \frac{5.2^6(2^6-1)B_3}{6!}x^4 + \cdots,$$

$$\operatorname{cosec}^2 x = \frac{1}{x^2} + \frac{1}{3} + \frac{3.2^4B_2}{4!}x^2 + \frac{5.2^6B_3}{6!}x^4 + \cdots$$

20. By putting $\phi = \pi x$ and taking derivatives, derive from equation (1), Art. 239, the series

$$\pi \cot \pi x = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + \frac{1}{x-3} + \cdots$$

21. By a similar method derive the series

$$\frac{\pi}{2}\tan \pi x = \frac{1}{1-2x} - \frac{1}{1+2x} + \frac{1}{3-2x} - \frac{1}{3+2x} + \cdots$$

22. By taking successive derivatives of the series obtained in Ex. 20 or of that obtained in Ex. 21, and then putting $x = \frac{1}{4}$, derive the following numerical series:

$$\begin{aligned}
\frac{\pi^2}{8} &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots, \\
\frac{\pi^3}{32} &= 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots, \\
\frac{\pi^4}{96} &= 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots, \\
\frac{5\pi^5}{1536} &= 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots, \\
\frac{\pi^6}{960} &= 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots.
\end{aligned}$$

The values of the successive derivatives for $\pi x = \frac{1}{4}\pi$ *are most readily found by the process of Art. 190. The first, third and fifth of these results may also be derived from Ex. 13 above.*

CHAPTER VII.

APPLICATION TO PLANE CURVES.

XXIV.

Tangent and Normal at a Given Point.

245. We have seen that, in the case of a plane curve referred to rectangular coordinates, if ϕ denotes the inclination of the curve at the point (x,y), and s the length of the curve as measured from some fixed point on it, we have

$$\tan\phi = \frac{dy}{dx}, \qquad \sin\phi = \frac{dy}{ds}, \qquad \cos\phi = \frac{dx}{ds}, \quad . \quad (1)$$

and

$$ds = \sqrt{(dx^2 + dy^2)}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

See Fig. 18, p. 91, in which the right-angled differential triangle is drawn.

246. If x_1, y_1 are the coordinates of a known point on the curve, the equation of the tangent at that point is found by giving to m in the general equation

$$y - y_1 = m(x - x_1)$$

the value of $\tan\phi$ at the point (x_1, y_1); hence it is

$$y - y_1 = \left.\frac{dy}{dx}\right]_{x_1, y_1} (x - x_1). \quad . \quad . \quad . \quad . \quad (1)$$

In like manner, the equation of the normal at the point (x_1, y_1) is found by giving to the direction ratio m the value $\tan(\frac{1}{2}\pi + \phi) = -\cot\phi$; hence it is

$$y - y_1 = -\frac{dx}{dy}\bigg]_{x_1, y_1}(x - x_1). \quad . \quad . \quad . \quad . \quad (2)$$

For example, in the case of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we have $\dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$, therefore the equation of the tangent at (x_1, y_1), a point of the curve, is

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1).$$

This equation may, by means of the equation

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

(which expresses that (x_1, y_1) is a point on the ellipse), be put in the form

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

Again, the equation of the normal is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1),$$

or

$$a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1.$$

Tangent and Normal at the Origin.

247. When the curve passes through the origin, we have seen, in Art. 172, that the value of $\frac{dy}{dx}$ at the origin is the same as that of $\left.\frac{y}{x}\right]_0$, and may be determined by simply equating to zero the terms of the lowest degree in the equation of the curve. It follows that the equation so found is itself the equation of the tangent at the origin, because in that line the value of the ratio $y:x$ is constant. Thus, in the example given in Art. 172, the tangent at the origin to the circle

$$x^2 + y^2 - 2x + y = 0$$

is the line

$$2x - y = 0.$$

The equation of the normal at the origin (in which m is the negative reciprocal of m in the tangent) is, in this case,

$$2y + x = 0.$$

Curves Touching one of the Axes at the Origin.

248. When, not only the absolute term, but one of those of the first degree is absent from the equation of the curve, it passes through the origin and there touches one of the coordinate axes. For example, the curve

$$x^3 + x^2y - 2xy^2 + x^2 - 2y = 0 \quad . \quad . \quad . \quad (1)$$

passes through the origin; and its tangent at that point is the line $y = 0$, that is to say, the axis of x.

In every such case, a process similar to that employed in Art. 172 gives the equation of a simple curve which has a much closer contact with the given curve than the tangent has. Thus, dividing equation (1) throughout by x^2, we may put it in the form

$$x + y - 2\frac{y^2}{x} + 1 - \frac{2y}{x^2} = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Now we already know that at the origin the ratio $\frac{y}{x} = 0$; therefore $\frac{y^2}{x} = 0 \times 0 = 0$, but $\frac{y}{x^2}$ takes the form $\frac{0}{0}$ and may have a finite value at the origin. Hence, putting $x = 0$ and $y = 0$, we have

$$1 - \frac{2y}{x^2}\bigg]_{0,0} = 0,$$

which gives, for the ratio in question, the value $\frac{1}{2}$.
Hence the simpler curve

$$x^2 - 2y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives the same value to this ratio, must approach very closely to the given curve (1) for small values of x, that is to say, in the neighborhood of the origin.

249. The simple auxiliary curve thus found is in any case readily constructed, and is said to give *the form of the given curve at the origin.* In the example above, it is a common parabola situated as in Fig. 1, p. 4. Since it lies above the axis of x to which it is tangent, we infer that the given curve also lies above the axis in the neighborhood of the origin.

In like manner, still supposing the given equation to contain no absolute term, but to contain the term in y, if the term of lowest degree not involving y contains x^n, we have an auxiliary equation, consisting of two terms only, which determines a finite value for the ratio $y : x^n$ at the origin. The corresponding curve determines the form of the given curve at the origin.

So also when the given curve touches the axis of x.

The Parabola of the nth Degree.

250. The general equation of the auxiliary curves considered above may be written in the homogeneous form,

$$a^{n-1}y = x^n. \qquad \text{. . . . . . .} \qquad (1)$$

The curve represented is called *the parabola of the nth degree.* Supposing a in equation (1) to be positive, the curve passes through the point (a, a), as well as through the origin. When $n > 1$, the curve touches the axis of x, and when $n < 1$, it touches the axis of y.

The following three diagrams represent forms which the curve takes for different values of n greater than unity. When n is a fraction, it is supposed to be reduced to its lowest terms.

Fig. 35 represents the general shape of the curve when n is an even integer, or a fraction having an even numerator and an odd denominator.

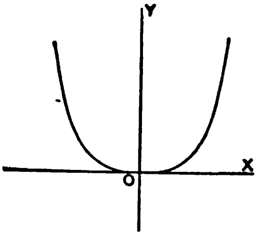

FIG. 35.

Fig. 36 represents the form of the curve when n is an odd integer or a fraction with an odd numerator and an odd denominator; the origin is in this case a point of inflection.

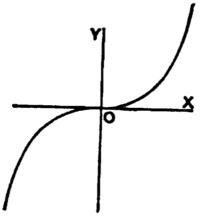

FIG. 36.

Fig. 37 represents the form of the curve when n is a fraction having an odd numerator and an even denominator. In this case, y is a two-valued function, and is imaginary when x is negative.

Fig. 35 is constructed for the parabola in which $n = 4$.

Fig. 36 is the *cubical parabola* in which $n = 3$.

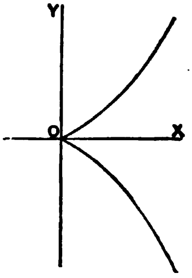

FIG. 37.

Fig. 37 is the *semi-cubical parabola* in which $n = \frac{3}{2}$; the equation being

$$a^{\frac{1}{2}}y = \pm x^{\frac{3}{2}},$$

or

$$ay^2 = x^3.$$

251. It will be noticed that the curve in each case consists of two like branches, symmetrically situated with respect either to an axis or to the origin as a centre. Since a was assumed positive in equation (1), one of these branches, in each diagram, is in the first quadrant.

The auxiliary equations, found as in Arts. 248 and 249, have the more general form

$$Ay^q + Bx^p = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where A and B may have either sign, and the positive integers p and q may be both odd, or one odd and one even. (If both are even, the equation will indicate an isolated point if A and B have the same sign, and will be decomposable, indicating more than one branch, if A and B have opposite signs.) The three diagrams, in different positions with respect to the axes, then give the forms which the given curve* may have at the origin. Thus Figs. 26 and 27, pp. 133 and 134, show cases in which $n < 1$, or $p < q$. The axis which is touched is in every case given by the term of lowest degree, and the two quadrants containing the branches are readily determined by the odd or even character of p and q. For instance, in $2x + y^4 = 0$, y can evidently have either sign, but x must be negative; therefore the branches lie in the second and third quadrants. Again, in $2x + y^3 = 0$, x and y must have opposite signs; hence the branches lie in the second and fourth quadrants.

Subtangents and Subnormals.

252. Certain lines connected with a curve and the coordinate axes, and varying with the point (x, y) of the curve, have received special names. The most important of these are *the subtangent* and *the subnormal*. At the point (x, y), P in Fig. 38, let the tangent and normal be drawn, cutting the axis of x in T and N, and the ordinate $PR = y$, then the segment TR is the subtangent, and RN is the subnormal. Hence, from the triangles TPR and PRN we have, for the subtangent,

$$TR = y \cot \phi = y \frac{dx}{dy},$$

and, for the subnormal,

$$RN = y \tan \phi = y \frac{dy}{dx}.$$

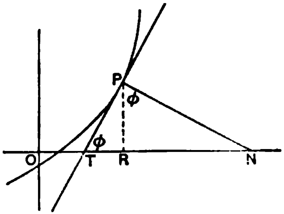

FIG. 38.

These formulæ give positive values

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[*] When p and q each exceed unity, the equation does not necessarily give the form at the origin of a curve in whose equation the terms $Ay^q + Bx^p$ are the lowest in degree, containing each a single variable. For example, if $3x^2 + 2y^3$ are the terms of this character, $3x^2 + 2y^3 = 0$ does not give the form at the origin if the given equation contains a term in xy. The complete criterion is best applied by means of the *Analytical Triangle*, as explained in works on Curve Tracing.

when the direction from T to R and from R to N respectively are to the right.

253. The segments PT and PN are sometimes called ***the tangent*** and ***the normal.*** Their values are

$$PT = y \operatorname{cosec} \phi = y\frac{ds}{dy},$$

$$PN = y \sec \phi \quad = y\frac{ds}{dx}.$$

In applying these formulæ and others involving ds, it must be remembered that equations (1), Art. 245, imply that ϕ indicates the direction in which ds is measured positively. Hence the diagram shows that PT when positive is to be measured from P in the direction $-\phi$, and PN when positive in the direction $\phi - 90°$.

For example, in the case of the curve

$$y = \sin x,$$

Fig. 12, p. 63, we have

$$dy = \cos x\, dx;$$

whence

$$ds^2 = (1 + \cos^2 x)dx^2,$$

and we write

$$ds = \sqrt{(1 + \cos^2 x)}dx.$$

Here, ds being taken with the same sign as dx, ϕ is the direction of the motion of a point moving toward the right. Substituting in the expression for the normal, we have

$$PN = \sin x \sqrt{(1 + \cos x)}.$$

This is positive when sin x, or y, is positive; accordingly, for a point above the axis of x, it is measured in the direction $\phi - 90°$, and for a point below in the direction $\phi + 90°$.

The Perpendicular from the Origin upon the Tangent.

254. If OQ in Fig. 39, the perpendicular upon the tangent PQ, be denoted by p, we have, from the triangles in the diagram (which is so drawn that x, y, sin ϕ and cos ϕ are positive),

$$p = x \sin \phi - y \cos \phi.$$

Substituting the values of sin ϕ and cos ϕ, this becomes

$$p = \frac{xdy - ydx}{ds} = \frac{xdy - ydx}{\sqrt{(dx^2 + dy^2)}}.$$

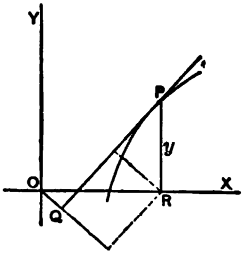

FIG. 39.

The figure shows that the direction of OQ (or p as drawn *from* the origin) is $\phi - 90°$ when positive.

Curve Tracing.

255. A curve whose equation is given is said to be *traced* when the general form of its various branches, and their position with respect to the coordinate axes, is determined. We notice, in the first place, certain forms of symmetry of which the occurrence will be indicated by the form of the equation. First, when it contains only even powers of one of the coordinates the curve is symmetrical to one of the axes. Thus the curve

$$x(x^2 + y^2) + a(x^2 - y^2) = 0 \quad . \quad . \quad . \quad . \quad (1)$$

is symmetrical to the axis of x; because, if the equation is satisfied by the point (x, y), it is also satisfied by the point $(x, -y)$ situated symmetrically to (x, y) with respect to this axis.

Again, if every term is of an even degree with respect to x and y jointly, or if every term is of an odd degree, the curve is symmetrical with respect to the origin as a centre. For example,

$$Ax^2 + Bxy + Cy^2 = D$$

is thus symmetrical, because, if (x, y) satisfies the equation, $(-x, -y)$ also satisfies it. This is in fact the equation of the conic with respect to the centre as origin; if $B = 0$, it is the conic referred to its axes, which is symmetrical to both axes and therefore also to the origin.

256. If the equation can be solved with respect to one of the variables, so as to make it an explicit function of the other, it is generally advantageous to do so. Thus, if the equation is put in the form $y = f(x)$, the curve becomes the graph of a known function, so that, by assigning values to x, we may determine as many points as we choose through which to draw a continuous curve.

The most important things to be determined are the limits of continuity, whether indicated by infinite values of y as illustrated in Fig. 3, p. 6, or by values of x on one side of which y is imaginary, as in Fig. 4, p. 7. Next to these come the values of x for which $y = 0$, giving points where the curve cuts the axis of x, and the values of y corresponding to $x = 0$ and to $x = \infty$ respectively.

257. As an illustration, let us take the curve represented by equation (1) of the preceding article, which is known as *the Strophoid*. This curve is symmetrical to the axis of x; solving its equation for y^2, we have

$$y^2 = \frac{x^2(x+a)}{a-x}. \quad . \quad . \quad (1)$$

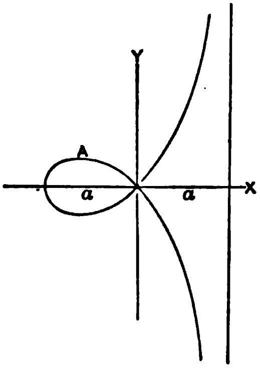

Fig. 40.

Here $y = 0$ when $x = -a$ and when $x = 0$: y is infinite when $x = a$, and is real only between the limits $x = \pm a$. Hence the general shape of the curve is that given in Fig. 40, consisting of a loop between $x = -a$ and $x = 0$ and infinite branches between $x = 0$ and $x = a$. The tracing indicates the existence of a maximum ordinate. This corresponds to the maximum value of y^2 in equation (1). Taking derivatives

$$2y\frac{dy}{dx} = \frac{(a-x)(2ax + 3x^2) + ax^2 + x^3}{(a-x)^2} = \frac{2x(a^2+ax-x^2)}{(a-x)^2};$$

hence y^2 is a maximum when $x^2 - ax = a^2$. The roots of this quadratic are $x = \frac{a}{2}(1 \pm \sqrt{5})$. The positive root is beyond the limits of real values of y; the negative root is about $-.6a$ and the corresponding value of y is almost exactly $.3a$. These are the coordinates of A in the diagram. The tangents at the origin are found, by the method of Art. 247, to be the lines $y = \pm x$, bisecting the angles between the axes.

258. The maxima and minima values of either coordinate are its limiting values when it is made the independent variable. Consider, for example, the curve

$$y^4 + x^2 - y^2 = 0,$$

which is symmetrical to both axes and therefore also to the origin as a centre. Solving for x,

$$x = \pm y \sqrt{(1 - y^2)}:$$

hence the limiting values of y are ± 1 (the corresponding points are those in which the curve cuts the axis of y), and x is real between the limits. The maximum value of x may be found either by the differential method, or as follows: Solving for y, we have

$$y^2 = \tfrac{1}{2} \pm \sqrt{(\tfrac{1}{4} - x^2)},$$

whence the limiting values of x are $\pm\frac{1}{2}$, and y is real between the limits. These are therefore the numerically greatest values of x, and the curve is limited to the rectangle between the lines $y = \pm 1$ and $x = \pm\frac{1}{2}$. The limiting values of x make $y^2 = \frac{1}{2}$, hence the curve touches the sides of the rectangle in the points $(0, \pm 1)$ and $(\pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{2})$. It passes through the origin at angles of 45° with the axes, and resembles in form a figure 8.

Points of Inflexion.

259. The curve considered above obviously has points of inflexion at the origin. In other cases, the form of the curve may indicate the existence of points of inflexion which, when the equation is solved for one of the variables, may be found by equating the second derivative to zero, see Art. 99. As an illustration, let us take the curve

$$y^2 = \frac{a^2x}{a - x},$$

which is symmetrical to the axis of x, and in which y is real only between the limits $x = 0$ and $x = a$. It cuts the axis of x only at the origin, where it touches the axis of y (as indicated by the double root when we put $x = 0$). Therefore $\tan\phi$ must be infinite at the origin and must at first decrease; but it is again infinite when $x = a$, because this value of x makes y infinite. There is therefore at least one point on the positive branch of the curve where the slope is a minimum, that is to say, a point of inflexion. The location of this point (of which the abscissa will be found to be $\frac{1}{4}a$), together with the slope of the curve at that point, will determine its form with considerable accuracy. The curve is known as *the Witch of Agnesi.*

The methods explained in the next section, which are applicable when the equation cannot be solved for either variable, are also often useful even when it can be so solved.

Examples XXIV.

1. In the case of the parabola of the nth degree

$$a^{n-1}y = x^n,$$

find the equations of the tangent and the normal at the point (a, a).

2. Find the equation of the tangent at any pointof the curve

$$x^n + y^n = a^n.$$

$$yy_1^{n-1} + xx_1^{n-1} = a^n.$$

3. Find the equation of the tangent at any point of the curve

$$x^m y^n = a^{m+n}.$$

$$nx_1y + my_1x = (m + n)x_1y_1.$$

4. Show that all the curves represented by the equation

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

(different values being assigned to n) have a common tangent at the point (a, b); find the equation of this tangent.

5. Show that the equation of the tangent to the curve

$$\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1,$$

at the point (x_1, y_1), is

$$a^{\frac{2}{3}}{x_1}^{\frac{1}{3}}y + b^{\frac{2}{3}}{y_1}^{\frac{1}{3}}x = a^{\frac{2}{3}}b^{\frac{2}{3}}{x_1}^{\frac{1}{3}}{y_1}^{\frac{1}{3}};$$

and, denoting the intercepts on the axes by x_0 and y_0, prove that

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1.$$

6. Given the curve

$$x^2 - 2y^2 - 4xy - x = 0,$$

show that the point $(1, -2)$ is on the curve, and find the equation of the tangent line at this point.

7. Find the subtangent and the subnormal of the parabola

$$y^2 = 4ax;$$

also the value of p in terms of x.

For the upper branch, $p = -\dfrac{x\sqrt{a}}{\sqrt{(a+x)}}$.

8. Find the subnormal of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

9. Prove that the normal to the catenary curve

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$$

(see Fig. 31, p. 217) is equal to $\frac{y^2}{c}$.

10. Trace the curve $y = x(x^2 - 1)$, finding the maximum ordinate and the slope at the intersections with the axis of x.

11. Trace the curve $y = x(1 - x)^2$, finding the maximum ordinate and point of inflexion.

12. Trace the curve $ay^2 = x(a - x)^2$.

13. Trace the curve $x^2y + a^2y - a^3 = 0$, finding points of inflexion.

14. Trace the curve $xy^2 = x^2 + y^2$.

15. Trace the curve $y^2 = x^3 - x^4$.

16. Trace the curve $y(x - a)^2 = a^2x$, finding a minimum ordinate and a point of inflexion.

17. Trace the curve $(y - x^2)^2 = x^5$.

18. Trace the curve $x(x^2 + y^2) = 2ay^2$, which is *the Cissoid of Diocles.*

19. Show that in the curve $y^2 = f(x)$ the abscissa of a point of inflexion will satisfy the equation

$$[f'(x)]^2 = 2f(x)f''(x).$$

20. The equation of *the Conchoid of Nicomedes* is

$$(x^2 + y^2)(x - a)^2 = b^2x^2:$$

trace the curve in the three cases when $b < a$, $b = a$, and $b > a$.

The maximum ordinate is $(b^{\frac{2}{3}} - a^{\frac{2}{3}})^{\frac{3}{2}}$; the abscissæ of the points of inflexion satisfy $x^3 - 3a^2x + 2a(a^2 - b^2) = 0$.

XXV.

Points at Infinity.

260. When the equation of a curve permits one or both of the coordinates to take an infinite value, that is to say, to increase without limit, the curve is said to have a ***point at infinity.*** If, while one of the coordinates becomes infinite, the other remains finite and ***has a definite limiting value***, there is a straight line, parallel to one of the axes, to which the point describing the curve approaches without limit as it recedes to infinity. This line is called an ***asymptote.*** We have had examples in Figs. 3, p. 6; 4, p. 7; 10, p. 57; 17, p. 71, etc., where a finite value of the variable regarded as independent gives an infinite value to the function; or else, as the independent variable becomes infinite, the function approaches without limit to a definite value.

In these cases, $\frac{dy}{dx}$ tends to one of the limits zero or infinity; and, if the point of contact of a tangent line recedes to infinity, the tangent approaches the asymptote as its limiting position; hence the asymptote is called ***the tangent at infinity.***

On the other hand, Fig. 12 exemplifies a case where, as x becomes infinite, neither y nor $\frac{dy}{dx}$ approaches a definite limit; there is then no asymptote and no definite tangent at infinity.

261. In the cases considered above, the point at infinity is said to be in the direction of one of the axes. The direction of the point at infinity is, of course, the direction of the line joining it to the origin. Hence, just as m is called the direction ratio of the line $y = mx$, so the value of the ratio $\frac{y}{x}$ at the infinite point gives the direction of the point at infinity.

When x and y become infinite simultaneously, the ratio $\left.\frac{y}{x}\right]_\infty$ takes the form $\frac{\infty}{\infty}$ and may have a finite value. If so, giving this value to m, the point at infinity is in the direction of the line $y = mx$. The ratio $y : x$ at infinity is the same for all parallel lines of the form $y = mx + b$, therefore parallel lines are said to pass through the same point at infinity. If an infinite branch of a curve has an asymptote in the direction $y = mx$ its equation will be of the form $y = mx + b$. It must not be inferred that an asymptote necessarily exists, since for this purpose it is necessary that b should admit of a finite value. See Art. 271.

262. To illustrate the method of finding the points at infinity for an algebraic curve, let us take the curve whose equation is

$$x^3 - xy^2 + ay^2 - a^2y = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Dividing through by x^3, we have

$$1 - \frac{y^2}{x^2} + \frac{a}{x}\frac{y^2}{x^2} - \frac{a^2}{x^2}\frac{y}{x} = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Putting $x = \infty$ and $y = \infty$, while assuming that the ratio $\left.\frac{y}{x}\right]_\infty$ has a finite value, we find

$$1 - \left.\frac{y^2}{x^2}\right]_\infty = 0,$$

whence $\left.\frac{y}{x}\right] = \pm 1$, which determines two points at infinity in the directions of the lines $y = \pm x$.

263. It will be noticed that, in this process, all the terms except those of the highest degree in the given equation disappear from the result, which is therefore the same as if the equation had consisted only of the group of terms of highest degree, namely, $x^3 - xy^2 = 0$. In fact, writing this equation in the factored form

$$x(x+y)(x-y) = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we see that the factors which, separately equated to zero, give its several roots constitute the equations of the lines through the origin in the direction of the several points at infinity. The root $x = 0$, in equation (3), corresponds to a point at infinity in the direction of the axis of y; for this point, the ratio $\frac{x}{y}$ is zero, or $\frac{y}{x}$ is infinite.

The number of points at infinity may, as in this case, equal the index of the degree of the equation, but cannot exceed it. If some of the roots of the equation are imaginary, there are fewer points at infinity, and when the degree is even there may be no real points at infinity.

The Equation of the Asymptote.

264. We proceed to determine the position of the asymptotes corresponding to the points at infinity determined by equation (3). For this purpose, we write the group of terms of the highest degree in the factored form. Then, for the asymptote corresponding to $x - y$, divide the equation of the curve through by the other factors, thus:

$$x - y = \frac{a^2y - ay^2}{x(x+y)}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, when a point recedes to infinity on this branch of the curve, x and y increase without limit, but with a limiting ratio of equality; thus the first member takes the illusory form $\infty - \infty$. But the second member, which takes the form ∞/∞, may be evaluated as in Art. 155. Thus, dividing both terms of the fraction by x^2, we have

$$x - y = \frac{\frac{a^2}{x}\frac{y}{x} - a\frac{y^2}{x^2}}{1 + \frac{y}{x}},$$

in which, putting $x = \infty$, we have, since $\left.\frac{y}{x}\right]_\infty = 1$,

$$x - y = -\tfrac{1}{2}a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We infer then that the distant points on the curve, both in the first and in the third quadrant, approach without limit to the corresponding points of the line represented by this equation. This line is therefore an asymptote.*

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* The value of the expression $x - y + \frac{1}{2}a$, which is zero for a point on the line (5), is for a point on the curve its vertical distance from the asymptote (below it when positive, and above it when negative), and this distance approaches zero as a limit.

265. It will be noticed that, in this process, only the terms of highest degree in the numerator (which are those of the next to the highest degree in the equation of the curve) can affect the result.

In practice, it is unnecessary to reduce the fraction in the second member to the complex form. Thus, in finding the asymptote corresponding to the factor $x+y$, we write

$$x+y=\left.\frac{-ay^2}{x(x-y)}\right]_{x=-y=\infty}=-\frac{a}{2}. \quad . \quad . \quad (6)$$

Again, corresponding to the factor x, we have

$$x=\left.\frac{-ay^2}{x^2-y^2}\right]_{y=\infty}=a; \quad . \quad . \quad . \quad . \quad . \quad (7)$$

since in this case $\left.\frac{x}{y}\right]_\infty = 0$

266. In this last case, it will be noticed that the asymptote depends solely upon the terms containing y^2 in equation (1), namely $-xy^2+ay^2$. Thus the absence of the term containing y^8 indicates a point at infinity in the direction of the axis of y, and then the equation of the asymptote is found by equating to zero the coefficient of y^2, that is to say, it is $-x+a=0$.

Tracing of Curves with Infinite Branches.

267. The construction of the asymptotes, when they exist, is of paramount importance in tracing the general form of a curve. For example, in the case of the curve

$$x^3-xy^2+ay^2-a^2y=0 \quad . \quad . \quad . \quad . \quad (1)$$

considered in the preceding articles, the three asymptotes, equations (5), (6) and (7), Arts. 264 and 265, are constructed as dotted lines in Fig. 41. These lines, together with a few actual points of the curve, such as its intersections with the coordinate axes, will generally serve to determine the shape of the several branches of the curve.

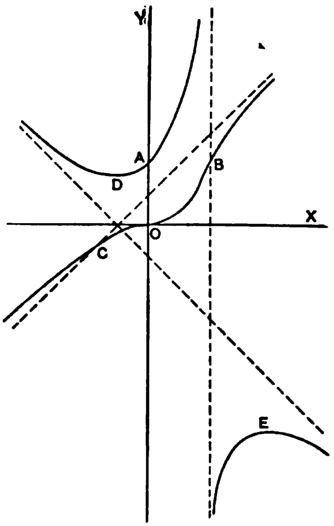

FIG. 41.

268. In the present case, putting $x = 0$ in equation (1), we have $ay^2 - a^2y = 0$, whence $y = 0$ and $y = a$, showing that the curve passes through the origin and through the point $(0, a)$, the point A in Fig. 41.

Putting $y = 0$ in equation (1), we have $x^3 = 0$, showing that the curve meets the axis of x only at the origin. The tangent at the origin is this axis, and, by the method of Art. 248, the form at the origin is given by

$$x^3 - a^2y = 0.$$

Here x and y have the same sign (see Art. 251); hence the branch passing through the origin has the form indicated in the diagram.

269. Since the curve is of the third degree, a straight line will in general cut it in three points. But special cases arise: for example, in the present case all three intersections with the axis of x coincide at the origin, because that axis is a tangent at a point of inflection. Again, the axis of y

cuts the curve in but two finite points, the third intersection being at infinity, because that axis is parallel to an asymptote.

Again, the asymptote itself, being a tangent at infinity, has but one finite or actual intersection with the curve. Accordingly, putting $x = a$ in equation (1), we have $a^3 - a^2y = 0$, giving the single root $y = a$ which determines the point (a, a), the point B in Fig. 41.

In like manner, the asymptote $x - y = -\frac{1}{3}a$ will be found to cut the curve in the single point $(-a, -\frac{1}{3}a)$, C in Fig. 41. It is clear that the other asymptote cuts the same branch of the curve, therefore the branch through A cannot cut either asymptote, and must approach the upper ends of these asymptotes in the manner indicated. Furthermore, the lower ends of these asymptotes must be approached by a third branch, as represented in the diagram.

Maximum and Minimum Coordinates.

270. The tracing of this curve shows that a point of minimum ordinate must exist in the branch through A, and one of maximum ordinate in the lower branch. To determine these we have, by differentiation of equation (1), Art. 267,

$$(3x^2 - y^2)dx - (2xy - 2ay + a^2)dy = 0.$$

It follows that $\frac{dy}{dx} = 0$ when

$$3x^2 - y^2 = 0, \qquad \text{or} \qquad y = \pm x\sqrt{3}.$$

Therefore the horizontal points of the curve are its intersections with these straight lines which pass through the origin.

Fig. 41 shows that the line $y = x\sqrt{3}$ (which makes an angle of 60° with the axis of x) has no other real intersection with the curve. But, putting $x = -\dfrac{y}{\sqrt{3}}$ in equation (1), we have the cubic

$$\frac{2y^3}{3\sqrt{3}} + ay^2 - a^2y = 0,$$

whence the three values of y are zero (corresponding to the origin) and the roots of the quadratic

$$2y^2 + 3\sqrt{3}ay = 3\sqrt{3}a^2.$$

These are $y = .77a$ and $y = -3.37a$, the former being the minimum ordinate at D and the latter the minimum negative ordinate at E in the lower branch.

Parabolic Branches.

271. When the factor of the group of terms of highest degree, indicating a point at infinity, is of the second degree, the process given in Art. 265 for the equation of the asymptote results in an infinite value for the second member,* showing that the point on the curve recedes indefinitely from the straight line drawn through the origin in the direction of the point at infinity. A branch of this kind, of which the common parabola presents the earliest instance, is said to be *parabolic.*

272. In the case of the curve

$$2x^2y + y^2 + 4x = 3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the single term of highest degree, $2x^2y$, indicates a point at infinity in the direction of each axis; and the case considered above arises with respect to that in the direction of the axis of y. That is to say, there is a branch upon which, as the point recedes indefinitely, $\frac{y}{x}$ becomes infinite, but x becomes infinite as well as y. In this case, x^2 will be found to have a finite ratio to y. For, equation (1) can be put in the form

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* Except when the group of terms of next highest degree contains the same factor, or is absent from the equation, in which case there are two parallel asymptotes. See Exs. 4, 7, etc., below.

$$\frac{2x^2}{y} + 1 + 4\frac{x}{y^2} = \frac{3}{y^2}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which, when y is infinite, reduces to

$$\left.\frac{2x^2}{y}\right]_\infty + 1 = 0.$$

It follows that the distant points of the curve approach the parabola

$$2x^2 + y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This parabola is said to give *the form at infinity* * of a branch of the curve. Since, in equation (3), x may have either sign, but y must be negative, the infinite branches in question lie in the third and fourth quadrants. These branches recede indefinitely from both axes, but they tend to parallelism to the axis of y.

Case in which one of the Axes is an Asymptote.

273. In the case of the point at infinity in the direction of the axis of x, indicated by the factor y in the highest term of equation (1), Art. 272, we have, by equating to zero the coefficient of x^2 (see Art. 266), the equation $y = 0$, indicating that the axis is itself the asymptote. When this is the case, it is easy to ascertain on which side of the asymptote the curve lies at either end. For, since y tends to the limit zero as x becomes infinite, xy or some other product of powers will tend to a finite limit when x is infinite. Now, dividing equation (1) by x, we have

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* In general, the parabola thus found, while it gives the form at infinity, is not *the asymptotic parabola*, or that to which the curve approaches indefinitely The equation of that parabola is found by evaluating the expression in the first member (in this case $2x^2 + y$) which takes the form $\infty - \infty$, exactly as in the process for the rectilinear asymptote, Art. 264. In the present case, however, the parabola (3) is asymptotic, as will be seen by multiplying equation (2) by y and then making y infinite.

$$2xy + \frac{y^2}{x} + 4 = \frac{3}{x};$$

and, for the point at infinity, this reduces to

$$xy + 2 = 0.$$

It follows that, for the distant points of the curve, x and y have opposite signs. Thus the branch approaching the right end lies below the axis, and that approaching the left end lies above it.

274. The curve

$$2x^2y + y^2 + 4x = 3, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whose infinite branches are considered in Arts. 272 and 273, is traced in Fig. 42. It intersects the axis of y in the two points $(0, \pm\sqrt{3})$, A and B in the figure; and the axis of x in the single point C, $(\frac{3}{4}, 0)$. The continuity of the branches now requires us to join the infinite branch in the second quadrant to A and then to C, and that in the third quadrant to B; but we

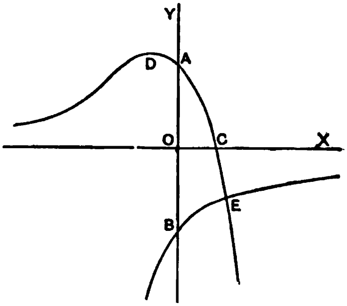


FIG. 42.

are left in doubt as to whether the other infinite branches should be joined to one another, or to *B* and *C* respectively.

275. To decide this point, we may search for maxima and minima coordinates; for a maximum and a minimum abscissa will exist if the first of these alternatives is the actual mode in which the branches join; whereas, in the other case, these will be replaced by a maximum and a minimum ordinate.

Differentiating equation (1) we have

$$\frac{dy}{dx} = \frac{-2(xy+1)}{y+x^2} = \frac{u}{v}.$$

To find horizontal points on the curve (for maximum ordinate) we must, as in Art. 127, combine the equation $u = 0$ with that of the curve. Eliminating x from equation (1) by means of the equation $xy + 1 = 0$, we have the cubic

$$y^3 - 3y - 2 = 0, \qquad \text{or} \qquad (y+1)^2(y-2) = 0.$$

To the root $y = 2$ corresponds $x = -\frac{1}{2}$, giving the point *D* in the figure, at which there is a maximum ordinate; but the double root $y = -1$ gives the point $(1, -1)$ which makes $v = 0$ also, so that $\frac{dy}{dx}$ takes the form $\frac{0}{0}$. This is, in fact, the example given in Art. 171 of that case, which indicates a double point; and the gradients of the two branches were there found to be $-2 \pm \sqrt{6}$, or -4.45 and $+0.45$, as shown at *E* in the diagram.

Examples XXV.

Find the asymptotes of the following curves:

1. $(x+a)y^2 = (y+b)x^2$.

$$x = -a, y = -b, y = x + b - a$$

2. $x^3 - 4xy^2 - 3x^2 + 12xy - 12y^2 + 8x + 2y + 4 = 0.$
$$x = -3,\ x = 2y,\ x + 2y = 6.$$

3. $(y - 2x)(y^2 - x^2) - a(y - x)^2 + 4a^2(x + y) - a^3 = 0.$
$$y = x,\ y + x = \tfrac{2}{3}a,\ y = 2x + \tfrac{1}{3}a.$$

4. $x^2y^2 + ax(x + y)^2 - 2a^2y^2 - a^4 = 0.$ $\qquad x = -2a,\ x = a.$

5. $x^7 - x^3y^4 + a^4y^3 - ax^2y^4 = 0.$
$$x = 0,\ x = -a,\ x + y = \tfrac{1}{4}a,\ x - y = \tfrac{1}{4}a.$$

6. $x^2(x - y)^2 - a^2(x^2 + y^2) = 0.$ $\qquad x = \pm a,\ y = x \pm a\sqrt{2}.$

7. $2x(x - y)^2 - 3a(x^2 - y^2) + 4(x - y)a^2 - 7a^3 = 0.$

8. Trace the curve $y^3 = x^2(x - a)$.

9. Trace the curve $x^3 - 2x^2y - 2x^2 - 8y = 0$, and show that $(-2, -1)$ is a point of inflexion.

10. Trace the curve $y^2(x - a) = x^2(x + a)$, and show that the origin is an isolated point or *acnode.*

11. Trace the curve $x^3 + y^3 - 3axy = 0$, which is known as *the Folium of Descartes.*

12. Trace the curve $2x^2y - x^2 + y^2 + 2x = 0$.

13. Trace the curve $x^3 - y^3 - x^2 - 2xy = 0$.

14. Trace the curve $x^3 + 2x^2y + xy^2 + a^2y = 0$, showing that it has parallel asymptotes.

15. Trace the curve $(x^2 - y^2)^2 - 4y^2 + 3y = 0$.

16. Trace the curve $x^3 - y^3 - x^2 + 2y^2 = 0$.

17. Trace the curve $x^3 + y^3 - x^2 - y^2 = 0$, and show that it is symmetrical to the line $x = y$.

18. Putting k in place of the second member of equation (1), Art. 274, trace the general form of the curve when $k < 3$ and when $k > 3$.

19. Trace the curve $x^4 - ax^2y + axy^2 + \frac{1}{4}a^2y^2 = 0$; show that there can be no negative values of x. The two branches meeting at the origin are said to form a *ramphoid cusp.*

20. Trace the curve $x^5 - 4ay^4 + 2ax^3y + a^2xy^2 = 0$.

XXVI.

Coordinates Expressed in Terms of a Third Variable.

276. The form of the rectangular equation of a curve sometimes suggests the expression of each of the coordinates x and y as explicit functions of a third or auxiliary variable. For example, the equation of the ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

suggests the employment of an auxiliary variable ψ, such that $\frac{x^2}{a^2}=\cos^2\psi$, whence $\frac{y^2}{b^2}=\sin^2\psi$. Hence we may put

$$x=a\cos\psi, \qquad y=b\sin\psi. \quad . \quad . \quad . \quad . \qquad (2)$$

Equations (2) have the advantage of expressing x and y as one-valued functions of ψ, so that each value of ψ distinguishes without ambiguity a single point of the curve. We may regard the point (x, y) as describing the whole curve, while ψ varies from 0 to 2π. On the other hand, when x is taken as the independent variable, y is a two-valued function, while x varies from $-a$ to $+a$.

277. We may now express the equation of a tangent to the ellipse in terms of the ψ of the point of contact. Thus, differentiating equations (2),

$$dx=-a\sin\psi d\psi, \qquad dy=b\cos\psi d\psi,$$

whence

$$\tan \phi = \frac{dy}{dx} = -\frac{b}{a}\cot \psi.$$

Now, substituting their values in terms of ψ for x_1 and y_1 in equation (1), Art. 246, we have, after reduction,

$$ay \sin \psi + bx \cos \psi = ab$$

for the equation of the tangent to the ellipse at the point $(a \cos \psi, b \sin \psi)$.

In like manner, for the normal at the same point, we have

$$y - b \sin \psi = \frac{a}{b}\tan \psi(x - a \cos \psi);$$

that is,

$$by \cos \psi - ax \sin \psi + (a^2 - b^2) \sin \psi \cos \psi = 0.$$

The Cycloid.

278. In the case of a number of important curves treated of in the following articles, the auxiliary variable is suggested by the definition of the curve as a geometrical locus. For example, the path described by a point in the circumference of a circle which rolls upon a straight line is called a *cycloid*. The curve consists of an unlimited number of branches corresponding to successive revolutions of the generating circle; a single branch is, however, usually termed a cycloid.

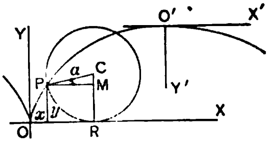


FIG. 43.

Let *O*, Fig. 43, the point where the curve meets the straight line, be taken as the origin, let *P* be the generating point of the curve, and denote the angle *PCR* by ψ. Since the arc *PR* is equal to the line *OR* over which it has rolled,

$$OR = PR = a\psi;$$

and, since $CM = a\cos\psi$, $PM = a\sin\psi$, we have

$$\left.\begin{aligned} x &= a(\psi - \sin\psi), \\ y &= a(1 - \cos)\psi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In these equations $\psi = 0$, gives the coordinates of *the cusp* of the curve situated at the origin, $\psi = \pi$ gives the coordinates of the highest point O' or *vertex*, $\psi = 2\pi$ corresponds to the next cusp or extremity of the first branch.

279. The employment of the two equations (1) is far more convenient than that of the single rectangular equation which results from eliminating ψ between them. For example, let it be required to find the direction of the motion of *P* and its linear velocity when the circle rolls uniformly with the angular rate ω. Differentiating equations (1),

$$dx = a(1 - \cos\psi)d\psi, \qquad dy = a\sin\psi d\psi,$$

therefore

$$\tan\phi = \frac{dy}{dx} = \frac{\sin\psi}{1 - \cos\psi} = \cot\tfrac{1}{2}\psi;$$

whence, taking ϕ in the direction of the motion when ψ increases,

$$\phi = 90 - \tfrac{1}{2}\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, squaring and adding we have

$$ds^2 = a^2(2 - 2\cos\psi)d\psi^2 = 4a^2\sin^2\tfrac{1}{2}\psi\,d\psi^2,$$

whence

$$ds = 2a\sin\tfrac{1}{2}\psi\,d\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Writing v for the linear velocity $\frac{ds}{dt}$, and ω for the angular velocity $\frac{d\psi}{dt}$,

$$v = 2a\omega\sin\tfrac{1}{2}\psi = 2V\sin\tfrac{1}{2}\psi,$$

where $V = a\omega$ is the linear velocity of the centre.

It readily follows from equation (2) that the chord RP of the circle is normal to the curve, and from equation (3) that the velocity of P in uniform rolling is proportional to RP.

280. The equations of the cycloid when in the inverted position are generally referred to the vertex O' as origin. In Fig. 43, O' is the point $(a\pi, 2a)$. Taking this as origin, and taking the opposite direction of the axis of y as positive, the new coordinates are $x' = x - a\pi$, $y' = 2a - y$, therefore

$$x' = a(\psi - \pi - \sin\psi) \qquad \text{and} \qquad y' = a(1 + \cos\psi);$$

but, in this case, it is more convenient to put ψ' for $\psi - \pi$; thus

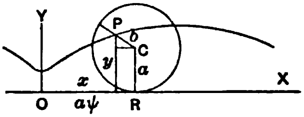
$$\left.\begin{aligned} x' &= a(\psi' + \sin\psi'), \\ y' &= a(1 - \cos\psi'). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1$$

In these equations, $\psi' = 0$ corresponds to the vertex at the origin, and $\psi' = \pm\pi$ corresponds to the cusps $(\pm a\pi, 2a)$.

The Prolate and Curtate Cycloids.

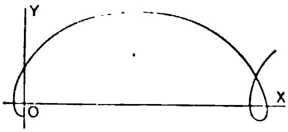
281. The curve described by a point in the plane of the rolling circle, either within or without the circle itself, is called a *trochoid*. Denoting by b the distance CP of the point from the centre, Fig. 44, and using the same notation as in Art. 278, so that OR is equal to the arc $a\psi$ subtending the angle RCP, we have

FIG. 44.



Prolate cycloid.

FIG. 45.



Curtate cycloid.

$$\left.\begin{aligned} x &= a\psi - b\sin\psi, \\ y &= \quad a - b\cos\psi. \end{aligned}\right\} \quad . \quad (1)$$

When $b<a$, the curve is the *prolate cycloid*, Fig. 44, and when $b>a$, the *curtate cycloid*, Fig. 45.

The Epicycloid and the Epitrochoid.

282. When a circle, tangent to a fixed circle externally, rolls upon it, the path described by a point in the circumference of the rolling circle is called an *epicycloid*.

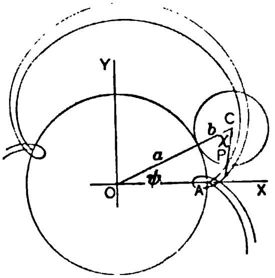


FIG. 46.

Taking the origin at the centre of the fixed circle, and the axis of x passing through A (one of the positions of P when in contact with the fixed circle), a, b, ψ and χ being defined by the diagram, we have, evidently,

$$a\psi = b\chi, \qquad \therefore \quad \chi = \frac{a}{b}\psi.$$

The inclination of PC to the axis of x is equal to $\psi + \chi$, that is to $\dfrac{a+b}{b}\psi$; the coordinates of P are found by subtracting the projections of PC on the axes from the corresponding projections of OC; hence

$$\left.\begin{aligned} x &= (a+b)\cos\psi - b\cos\frac{a+b}{b}\psi, \\ y &= (a+b)\sin\psi - b\sin\frac{a+b}{b}\psi. \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

283. If the describing point is taken on the radius CP at a distance c from the centre C, the curve described is called an *epitrochoid*. (When $c > b$, this is a looped curve as in Fig. 46.) Hence the equations of the epitrochoid are found by replacing the projections of b in equations (1) by those of c; thus they are

$$\left.\begin{aligned} x &= (a+b)\cos\psi - c\cos\frac{a+b}{b}\psi, \\ y &= (a+b)\sin\psi - c\sin\frac{a+b}{b}\psi. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

In equations (1), the axis of x passes through a cusp, and in equations (2), through one of the points *nearest* to the origin. If we change the sign of c, we have

$$\left.\begin{aligned} x &= (a+b)\cos\psi + c\cos\frac{a+b}{b}\psi, \\ y &= (a+b)\sin\psi + c\sin\frac{a+b}{b}\psi, \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

for the curve described by a point on the radius PC produced through C. Thus equations (3) are those of the epitrochoid when a *vertex*, or one of the points *farthest* from O, is situated upon the axis of x. If $c = b$, they become the equations of the epicycloid under the same circumstances.

The Hypocycloid and the Hypotrochoid.

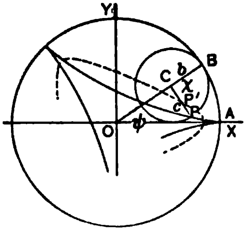


FIG. 47.

284. When the rolling circle has internal contact with the fixed circle, the curve generated by a point on the circumference is called a *hypocycloid*, whether the radius of the rolling circle be greater or less than that of the fixed circle. Curves generated by points on the radius, either within or without the circumference of the rolling circle, are called *hypotrochoids*.

Adopting the notation used in deducing the equations of the epitrochoid, we have (see Fig. 47)

$$OC = a - b, \quad \text{and} \quad \chi = \frac{a}{b}\psi.$$

The inclination of CP to the negative direction of the axis of x (an acute angle in the diagram) is

$$\chi - \psi = \frac{a-b}{b}\psi;$$

hence the equations of the *hypocycloid* are

$$\left.\begin{aligned} x &= (a-b)\cos\psi + b\cos\frac{a-b}{b}\psi, \\ y &= (a-b)\sin\psi - b\sin\frac{a-b}{b}\psi. \end{aligned}\right\} \quad . \; . \; . \; . \qquad (4)$$

In like manner, the equations of the hypotrochoid described by the point P' at a distance c from the centre are

$$\left.\begin{aligned} x &= (a-b)\cos\psi + c\cos\frac{a-b}{b}\psi, \\ y &= (a-b)\sin\psi - c\sin\frac{a-b}{b}\psi. \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

285. The equations of the epicycloid become those of the hypocycloid by changing the sign of b. Compare equations (1) and (4). So also equations (3) become equations (5) without changing the sign of c; because, in equations (5) as well as in equations (3), $\psi = 0$ gives one of the points *farthest* from the origin (see the dotted curve in Fig. 47).

These curves may all be included under a common definition or mode of generation. For, in Figs. 46 and 47, the point C describes a circle whose radius is $R = a + b$, b being negative in Fig. 47. At the same time, P describes a circle whose radius is c, about the moving point C as a centre. The rates of rotation of the radii R and c have a constant ratio, but in Fig. 47 the directions are opposite. Now putting in equations (3), Art. 283,

$$a + b = R, \qquad \frac{a+b}{b} = m,$$

we have

$$\left.\begin{aligned} x &= R\cos\psi + c\cos m\psi, \\ y &= R\sin\psi + c\sin m\psi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which m is negative when b is negative and numerically less than a, as is the case in Fig. 47.

In this point of view, the curves are called *epicyclics*.

286. It will be noticed that c and R in equations (6) may be interchanged, so that c becomes the radius of the fixed circle and R that of the one with moving centre, the relative rate of rotation being now $1/m$. This may be shown geometrically by means of a jointed parallelogram $OCPQ$, of which the sides OC and OQ (of lengths R and c) revolve about the fixed point O with rates of rotation having a constant ratio.* The opposite sides, being parallel to OC and OQ respectively, revolve at the same rates about the moving centres. Thus P describes the epicyclic, and the order in which R and c are taken is immaterial.

287. The relations in Art. 285 between the constants which occur in the form (3) and in the form (6) give

$$b = \frac{R}{m}, \qquad a = \frac{m - 1}{m}R, \qquad c = c, \quad . \quad . \quad (1)$$

for the reduction from the latter form to the form (3).

From what is shown in the preceding article it follows that, when a curve is given as an epitrochoid or hypotrochoid, there is a second method of generating it as such. For, after the equation is reduced to the form (6), we may interchange R and c (changing m to $1/m$), that is, put

$$R' = c, \qquad c' = R, \qquad m' = \frac{1}{m},$$

and then find constants a', b' and c' for a new expression in the form (3) by means of equations identical in form with equations (1). The resulting values will be found to be

$$a' = -\frac{ac}{b}, \qquad b' = \frac{c(a+b)}{b}, \qquad c' = a + b. \quad . \quad (2)$$

---

* See Fig. 48, in which the initial position of OC and OQ is in the axis of x and the angular rate of OQ is three times that of OC.

288. The following relation between the radii a, b, a' and b' is noteworthy. The equations above give

$$\frac{b'}{a'} = -\frac{a+b}{a}, \quad \text{or} \quad \frac{b'}{a'} + \frac{b}{a} = -1.$$

This shows that when a and b have the same sign (as in the epitrochoid), b' is opposite in sign to a', and is numerically the greater. Therefore the epitrochoid can be generated as a hypotrochoid in which the radius of the rolling is greater than that of the fixed circle.

On the other hand, when b is negative and numerically less than a, b' is negative and numerically less than a', the curve is a hypotrochoid with rolling circle smaller than the fixed circle in each mode of generation, and the numerical sum of the ratios is unity.

289. As an example of the double mode of generation, let us take the epitrochoid in which

$$a = 2, \quad b = 1 \quad \text{and} \quad c = \frac{3}{4}$$

in the form (3), Art. 283, which we have taken as the standard, the positive sign of c indicating that in the initial position c is measured away from the origin. The equations are

$$\left.\begin{aligned} x &= 3\cos\psi + \frac{3}{4}\cos 3\psi, \\ y &= 3\sin\psi + \frac{3}{4}\sin 3\psi, \end{aligned}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

so that, in the notation of Art. 285,

$$R = 3, \qquad c = \frac{3}{4}, \qquad m = 3.$$

Now equations (2), Art. 287, give

$$a' = -1\tfrac{1}{3}, \qquad b' = 2\tfrac{1}{4}, \qquad c' = 3,$$

for the constants in the second mode of generation, in which the curve is (in accordance with Art. 288) a hypotrochoid.

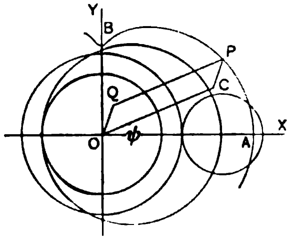


Fig. 48.

In Fig. 48, the initial positions of the rolling circles (corresponding to $\psi = 0$) in each mode of generation are shown, together with the position of the parallelogram of Art. 286 for a value of ψ about 25°.

This particular curve is symmetrical to both axes, because the point B nearest to the origin falls upon the axis of y.

Algebraic Forms of the Equations.

290. When the radii of the rolling and fixed circles are commensurable, the points of the circumferences which were originally in contact will again be in contact after a certain number of revolutions. In this case, the curve will begin to repeat itself, so that it consists of a finite number of branches. It will then admit of an algebraic equation, which is the result of eliminating ψ from its two equations.

For example, if $a = 2b$, the equations of the hypotrochoid, (5), Art. 284, become

$$\left.\begin{aligned} x &= (b + c)\cos\psi, \\ y &= (b - c)\sin\psi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which, see Art. 276, are the equations of the ellipse

$$\frac{x^2}{(b+c)^2} + \frac{y^2}{(b-c)^2} = 1. \quad . \quad . \quad . \quad . \quad (2)$$

Thus the hypotrochoid becomes an ellipse when the rolling circle is one-half the size of the fixed circle. Putting $c = b$ in equations (1), we have $y = 0$, showing that the corresponding hypocycloid is a straight line; that is to say, every point in the circumference of the rolling circle describes a diameter of the fixed one.

291. In the cycloidal cases, in which $c = b$, the same fixed circle serves in each of the modes of generation, and the cusps are situated upon it. In the cases now under consideration (the radii being commensurable), the cusps divide the circumference into equal parts. The epicycloids and hypocycloids may, in these cases, be distinguished by specifying the number of cusps, say r, together with, if necessary, the number of rth parts of the circumference covered by one branch. If more than one segment is thus covered, the branches cross one another.

In the case of the proper hypocycloids, which lie within the fixed circle, the sum of the radii of the rolling circles in the two modes of generation must, by Art. 288, be equal to a. Thus there is but one three-cusped hypocycloid, and in it the value of b is either $\frac{1}{3}a$ or $\frac{2}{3}a$; but there are two five-cusped hypocycloids, one of which is generated when $b = \frac{1}{5}a$ or $\frac{4}{5}a$, and is uncrossed, while the other, which is crossed or has double points, is generated when $b = \frac{2}{5}a$ or $\frac{3}{5}a$.

Again, if $b = a$, we have a one-cusped epicycloid which simply surrounds the fixed circle. (This curve is known as the *cardioid* and will be discussed later under another definition.) But if $b = 2a$, we have an epicycloid with a single cusp, one branch of which enwraps the fixed circle twice.

The Four-cusped Hypocycloid or Astroid.

292. The four-cusped hypocycloid may be generated by a rolling circle whose radius is $\frac{1}{4}$ that of the fixed circle, as indicated in Fig. 49, or by one whose radius is $\frac{3}{4}$ of a. Putting $b = \frac{1}{4}a$ in equations (4), Art. 284, we have

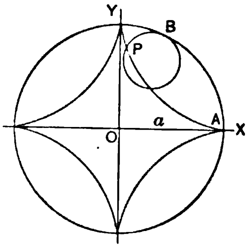


FIG. 49.

$$\left.\begin{aligned} x &= \frac{3}{4}a\cos\psi + \frac{1}{4}a\cos 3\psi, \\ y &= \frac{3}{4}a\sin\psi - \frac{1}{4}a\sin 3\psi. \end{aligned}\right\}$$

By means of the formulæ

$$\begin{aligned} \cos 3\psi &= 4\cos^3\psi - 3\cos\psi, \\ \sin 3\psi &= 3\sin\psi - 4\sin^3\psi, \end{aligned}$$

these reduce to

$$\left.\begin{aligned} x &= a\cos^3\psi, \\ y &= a\sin^3\psi. \end{aligned}\right\}$$

Eliminating ψ, we have

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

an equation which, when freed from radicals, is found to be of the sixth degree. This curve is sometimes called *the Astroid.*

Employment of m as an Auxiliary Variable.

293. When a curve whose rectangular equation is given has a multiple point at the origin, it is frequently convenient to express x and y in terms of m, where $y = mx$. If the curve is of the nth degree and has r branches passing through the origin, the straight line whose equation is $y = mx$ cuts the curve r times at the origin, and can therefore cut it in no more than $n - r$ other points. In fact, if we substitute $y = mx$ in the equation, every term in the result will contain x^r or a higher power of x. Dividing by x^r we then have an equation of the $(n - r)$th degree for x in terms of m.

294. For example, given the curve

$$x^4 - 3axy^2 + 2ay^3 = 0,$$

which is of the fourth degree, and has a triple point at the origin. Putting $y = mx$, we have

$$x^4 - (3am^2 - 2am^3)x^3 = 0,$$

and, rejecting the factor x^3 which gives three roots equal to zero,

$$x = 3am^2 - 2am^3;$$

whence

$$y = 3am^3 - 2am^4.$$

Thus x and y are, in this case, one-valued functions of m; and, by giving particular values to m, we may determine as many points as we please upon the curve. The values of m which make x and y vanish determine the tangents at the origin. They are, in this case, $m = 0$ and $m = \frac{3}{2}$. It is clear that, as m increases from 0 to $\frac{3}{2}$, the line $y = mx$ turns about the origin, and the point (x, y) upon it describes a loop of the curve returning to the origin, see Fig. 50. As m increases from $\frac{3}{2}$ to ∞, x and y

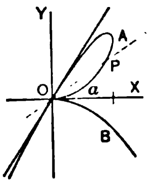


FIG. 50.

pass from 0 to $-\infty$ the point (x, y) describing the branch in the third quadrant. Finally, as m changes sign and passes from $-\infty$ to 0, x becomes positive and y negative, and the point (x, y) describes the branch in the fourth quadrant returning to the origin.

295. The maximum values of x and y occurring in the loop may be determined from their expressions in terms of m. Equating to zero the derivative of x, we thus find $m = 1$ and $m = 0$; the former gives the point (a, a) (A in the diagram) at which x is a maximum, and the latter the origin at which x in the cusp is a minimum of the variety shown in Fig. 27, Art. 133.

In like manner, from the derivative of y, we find $m^2 = 0$ and $m = \frac{9}{8}$; the former corresponds to a double root for which the derivative does not change sign, and the latter gives the point of maximum ordinate at about $(0.95a, 1.07a)$.

These values determine the loop with considerable accuracy. The branch in the fourth quadrant is equally well determined by the point B for which $m = -\frac{1}{2}$, namely, $(a, -\frac{1}{2}a)$, and the slope at that point, as given by the value of dy/dx.

296. The method is equally applicable when the origin is an isolated point or *acnode*. For example, the symmetrical curve

$$(x^2 + y^2)^2 = 4x^2 + y^2$$

gives

$$x^2 = \frac{4 + m^2}{(1 + m^2)^2},$$

$$y^2 = \frac{4m^2 + m^4}{(1 + m^2)^2}.$$

Here x and y cannot become zero, nor can they become infinite. The value $m = 0$ gives the points $(\pm 2, 0)$ on the axis of x, and $m = \infty$ gives $(0, \pm 1)$ on the axis of y. Treating y^2 as a function of m^2, it is readily shown that $m = \pm \sqrt{2}$ gives points of maximum ordinates at $(\pm \frac{1}{3}\sqrt{6}, \pm \frac{2}{3}\sqrt{3})$.

Examples XXVI.

1. The locus of the point M in Fig. 43 was called by Roberval "the companion to the cycloid." Show that it is a curve of sines (see Fig 12, p. 63) symmetrical to the centre of the rectangle OO' and therefore bisects its area.

The area between the two curves regarded as generated by the variable line PM parallel to OX is readily perceived to be equal to the area of the semicircle, which is generated by this line when the point M describes a diameter of the circle regarded as fixed. In this way Roberval proved that the area of the cycloid is three times that of the generating circle.

2. Prove that in the trochoid, as well as in the cycloid, the line PR is a normal and is proportional to the actual velocity of P in uniform rolling.

3. Determine the ordinate of the point of inflexion in the prolate cycloid.

$$y = \frac{a^2 - b^2}{a}.$$

4. Using the general equations (6), Art. 285, show that points of inflexion occur when

$$\cos(m - 1)\psi = -\frac{R^2 + m^2c^2}{m(m + 1)cR},$$

and hence show that the epitrochoid has points of inflexion when the numerical value of c lies between b and $\dfrac{b^2}{a + b}$, and similarly for the hypotrochoid.

5. Derive the algebraic equation of the two-cusped epicycloid with cusps on the axis of x. $4(x^2 + y^2 - a^2)^3 = 27a^4y^2$.

6. Show that in the four-cusped hypocycloid the auxiliary angle ψ is the inclination of the tangent at (x, y) to the negative direction of the axis of x; find also the value of the perpendicular from the origin.

$$p = a \sin \psi \cos \psi.$$

7. Show that in the four-cusped hypocycloid the intercept of the tangent between the axes is constant; also that the point of contact and the foot of the perpendicular are equally distant from opposite extremities of this line.

8. Determine the value of p for the epicycloid.

$$p = (a + 2b) \sin \frac{a\psi}{2b}.$$

9. Trace the curve $y^4 - x^4 + 2axy^2 = 0$.

10. Trace the curve $x^5 + y^5 - 5ax^2y^2 = 0$.

11. Trace the curve $x^5 + y^5 - 5ax^3y = 0$.

12. Trace the curve $x^5 + y^5 - 2a^3xy = 0$.

13. Trace the curve $x^3 - y^3 + (2y - x)^2 = 0$.

14. Trace the curve $y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0$, which has been called "*la courbe du diable.*" Determine maximum and minimum values of x.

15. Trace the curve $x^4 + y^4 + 6ax^2y - 8ay^3 = 0$.

16. Trace the curve $x^5 + y^4 - 5x^2y - 3xy^2 = 0$, determining points by putting $m = 1$, and finding the slope at those points.

17. Trace the curve $x^4 - ax^2y - axy^2 + a^2y^2 = 0$, showing that the line $y = mx$ touches the curve when $m = 1$ and when $m = -3$. Find a maximum value of x by solving for y.

18. Trace the curve $x^4 - 2a^2x^2 - 2ay^3 + 3a^2y^2 = 0$, showing that it has nodes at points corresponding to $m = \pm 1$.

XXVII.

Polar Equations.

297. When the equation of a curve is given in polar coordinates, we shall assume that it is solved for r, that is to say, given in the form $r = f(\theta)$. Let s be the length of an arc of the curve measured in the direction in which θ increases. Then, as θ increases, the point P in Fig. 51 moves with the velocity $\dfrac{ds}{dt}$. Let PT, a portion of the tangent line, represent ds; then, producing r, let the rectangle PT be completed, and let ψ denote the angle TPS; that is, the angle between the *positive directions* of r and s. The resolved velocities of P along and perpendicular to the radius vector are $\dfrac{dr}{dt}$ and $\dfrac{rd\theta}{dt}$, the latter being the velocity which P would have if r were constant; that is, if P moved in a circle described with r as a radius. Hence we have

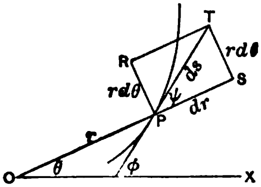


FIG. 51.

$$PS = dr \qquad \text{and} \qquad PR = rd\theta.$$

From the triangle PST, we derive

$$\tan\psi = \frac{rd\theta}{dr}, \qquad \sin\psi = \frac{rd\theta}{ds}, \qquad \cos\psi = \frac{dr}{ds}, \quad . \quad (1)$$

and

$$ds^2 = dr^2 + r^2 d\theta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In accordance with the assumption that ds has the sign of $d\theta$, we write

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and we infer from the second of equations (1) that the value of ψ will always be either in the first or in the second quadrant.

The first of equations (1) is equivalent to

$$\cot\psi = \frac{dr}{rd\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

298. It is frequently convenient to employ in place of the radius vector its reciprocal, which is usually denoted by u; then

$$r = \frac{1}{u}, \quad \text{and} \quad dr = -\frac{du}{u^2}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Making these substitutions, equations (3) and (4) give, in terms of u and θ,

$$\frac{ds}{d\theta} = \frac{1}{u^2}\sqrt{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]} \quad . \quad . \quad . \quad . \quad (6)$$

and

$$\cot\psi = -\frac{du}{ud\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Polar Subtangents and Subnormals.

299. Let a straight line perpendicular to the radius vector be drawn through the pole, and let the tangent and the normal meet this line in T and N respectively; then the projections of PT and PN upon this line, that is OT and ON, are called respectively the ***polar subtangent*** and the ***polar subnormal.*** In Fig. 52, $OPT = \psi$; whence

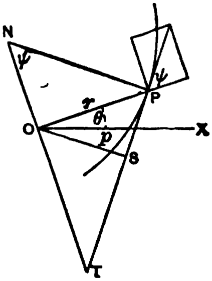


FIG. 52.

$$OT = r\tan\psi = r^2\frac{d\theta}{dr} = -\frac{d\theta}{du},$$

and

$$ON = r\cot\psi = \frac{dr}{d\theta} = -\frac{du}{u^2 d\theta}.$$

Fig. 52 shows that the value of OT is positive when its direction is $\theta - 90°$; that of ON is, on the other hand, positive when its direction is $\theta + 90°$.

The Perpendicular from the Pole upon the Tangent.

300. Let p denote the perpendicular distance from the pole to the tangent; then, from Fig. 52, we obtain

$$p = r\sin\psi = r^2\frac{d\theta}{ds} = \frac{r^2}{\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}}. \quad . \quad . \quad (1)$$

These expressions give positive values for p, because $\frac{ds}{d\theta}$ is assumed to be positive, and Fig. 52 shows that p has the direction $\phi - 90°$; ϕ being the angle which the positive direction of s makes with the initial line.

Equation (1) may be written in the form

$$\frac{1}{p^2} = \frac{ds^2}{r^4 d\theta^2},$$

and, transforming by the formulæ of Art. 298, we have

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Critical Points.

301. The critical points of a curve with reference to polar coordinates (analogous to the horizontal and vertical points of Arts. 127 *et seq.*) are those for which $\psi = 0$, and those for which $\psi = 90°$, while ' has a finite value. In the first case, the radius vector is tangent to the curve. As the moving point P describing the curve passes through such a point, dr has a finite value while $d\theta = 0$. Both the polar subtangent and the perpendicular p then vanish (see Arts. 299 and 300). Unless the point is at the same time a point of inflexion, θ has a limiting value such that, as θ passes through it, two real values of r become equal and then imaginary.

In the second case, when $\psi = 90°$, the radius vector is normal to the curve. We now have $p = r$, while the subtangent is infinite and the subnormal vanishes. The radius vector will, in this case, generally have either a maximum or a minimum value.

Zero Values of r.

302. Let $r = 0$ when θ takes the value θ_1; then, as θ passes through the value θ_1, the point describing the curve reaches the pole moving in the direction of the straight line whose inclination is θ_1. Accordingly, the equations of Art. 297 show that $\psi = 0$ or $180°$, and $ds = \pm dr$. In general, r will be found to pass through the value zero and become negative as θ passes through the value θ_1, so that the curve lies as usual upon one side of the tangent line.

As an illustration, let us take the curve

$$r = a \cos \theta \cos 2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Here $r = 0$ when $\cos \theta = 0$ and when $\cos 2\theta = 0$, that is, when $\theta = 90°$, $45°$ or $135°$. The dotted lines in Fig. 53 are the tangents at the pole. When $\theta = 0$, $r = a$. Hence the generating point, starting from A, describes the half loop in the first quadrant while θ increases from 0 to $45°$. When θ passes $45°$, r becomes negative, as indicated in the diagram, but it returns to zero when $\theta = 90°$, the point (r, θ) describing the loop situated in the third quadrant. As θ passes $90°$, r again becomes positive and the loop in the second quadrant is described. Finally, while θ passes from $135°$ to $180°$, r is again negative, and the point A is reached with the values $r = -a$, $\theta = 180°$.

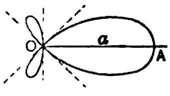


FIG. 53.

In this example, the change of θ to $\theta + \pi$, in equation (1), changes the sign, but not the value of r; and, since (r, θ) and $(-r, \theta + \pi)$ represent the same point, the curve repeats itself when θ varies from π to 2π, from 2π to 3π and so on.

303. The maximum positive and negative values of r will be found to correspond to $\theta = 0$ and $\theta = \pm \tan^{-1}5$. But the form of the loops is better determined, in this case, by the maximum value of the ordinate y when the initial line is taken as the axis of x. From equation (1), we have

$$y = r \sin\theta = a \sin\theta \cos\theta \cos 2\theta = \tfrac{1}{4}a \sin 4\theta.$$

The maximum positive and negative values occur when $4\theta = \frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, etc., that is when $\theta = \frac{1}{8}\pi$, $\frac{3}{8}\pi$, $\frac{5}{8}\pi$ and $\frac{7}{8}\pi$, and the numerical value of the maximum is in each case $\frac{1}{4}a$.

The Lemniscate.

304. The curve whose polar equation is

$$r^2 = a^2 \cos 2\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

known as *the Lemniscate of Bernoulli,* will serve to illustrate the case in which the value of θ which makes $r = 0$ is also a limiting value of θ, Art. 301. In equation (1), putting $\cos 2\theta = 0$, we have $\theta = 45°$, and $\theta = 135°$ for the values which make $r^2 = 0$. When θ varies from 0 to 45°, r is a two-valued function having numerically equal positive and negative values. These values decrease from the initial value $\pm a$ to zero. Thus the two generating points, starting from A and B, Fig. 54, describe the half-loops in the first and third quadrants, and meet at the origin, *forming a point of inflexion.* As θ passes through 45°, the values of r become imaginary, and so remain for all values between 45° and 135°; the rest of the curve being described when θ varies from 135° to 180°.

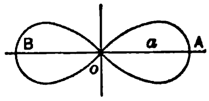


FIG. 54.

Or we may regard the whole curve as described while θ varies continuously from $-\frac{1}{4}\pi$ to $+\frac{1}{4}\pi$, the whole right-hand loop then corresponding to positive values of r.

305. In finding the maximum ordinates, we may put the function $y^2 = r^2 \sin^2\theta$ a maximum. Thus, from equation (1),

$$\sin^2\theta \cos 2\theta = \text{a maximum};$$

whence

$$2 \sin\theta \cos\theta \cos 2\theta - 2 \sin^2\theta \sin 2\theta = 0,$$

or

$$\sin\theta \cos 3\theta = 0.$$

The root $\sin\theta = 0$ makes y^2 (but not y) a minimum; but $\cos 3\theta = 0$ gives $\theta = \pm 30°$ for the vectorial angles of the maximum ordinates. The corresponding values of y are $\pm \frac{1}{4}a\sqrt{2}$.

Polar Equations involving only Trigonometric Functions of θ.

306. When, in the polar equation $r = f(\theta)$, only trigonometric functions of angles commensurable with θ occur, r will be a *periodic function* of θ and the period will be commensurable with 2π. Thus the period of $r = a \cos \frac{1}{2}\theta$ is 4π because adding 4π to θ is equivalent to adding 2π to $\frac{1}{2}\theta$. In other words, the radius vector returns at $\theta = 4\pi$, for the first time, to its initial value, a. Since this period 4π is a multiple of 2π, the generating point returns to its initial position, and the curve is completed when the vectorial angle has made two complete revolutions.

Again, the period of $r = a \cos 2\theta$ is π, because adding π to θ adds 2π to 2θ, that is r returns to its initial value a

when $\theta = \pi$. But the generating point does not return to its original position until $\theta = 2\pi$ when the curve is completed.

In each of these examples, the student should trace the curve, noting how the generating point completes its circuit with alternately positive and negative values of r. In each case, the loops so formed are similar; and in the second case (there being two corresponding to values of each sign, because the period of r is π), the curve consists of four equal loops.

307. In general, the curve is completed with n revolutions of the vectorial angle, where n is the least integer which makes $2n\pi$ a multiple of the period of r. But, when n is an odd number, it may happen that the curve will be completed when $\theta = n\pi$; namely, when the addition of $n\pi$ to θ changes the sign but not the numerical value of r. We have had an example in the curve of Art. 302, in which the period of r is 2π, but the addition of π to θ changes r to $-r$, hence the curve is completed when $\theta = \pi$.

Again, in $r = a \cos 3\theta$, the period of r is $\frac{2}{3}\pi$, and 2π is a multiple of this period. But the addition of π to θ changes the sign of r, hence the curve is completed with three equal loops when $\theta = \pi$; and, when θ varies from π to 2π, these loops are repeated with values of r opposite in sign.

Again, for the curve $r = a \cos \frac{1}{3}\theta$ the period of r is 6π, so that $n = 3$; but, for the same reason as in the preceding examples, the curve is completed when $\theta = 3\pi$.

The Limaçon of Pascal.

308. The curve obtained by producing the radius vector of a given curve by a fixed amount is called a *protraction* of the given curve with respect to the pole. The protraction of a circle with respect to a point on its circumference is a curve named by Pascal the *Limaçon*. Taking the diameter through the given point O on the circumference, Fig. 55, as the initial line, and denoting the radius $O'Q$ by a, the polar equation of the circle whose radius is a is

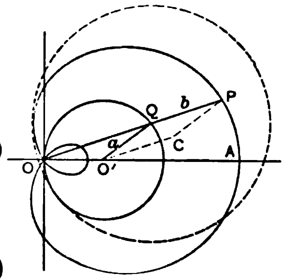


FIG. 55.

$$OQ = 2a \cos\theta. \quad . \quad . \quad (1)$$

Hence, if b is the constant amount of protraction,

$$r = 2a \cos\theta + b \quad . \quad . \quad (2)$$

is the equation of the limaçon.

The equation

$$r = 2a \cos\theta - b \quad . \quad . \quad (3)$$

represents the same curve. For, if in equation (2) we add π to θ, we obtain

$$r = -2a \cos\theta + b = -(2a \cos\theta - b),$$

the negative of the value corresponding to θ in equation (1); thus $\theta + \pi$ in equation (2) gives the same point that θ gives in equation (3).

309. If we join Q, the moving point on the circumference of the circle, with its centre O', the angle $QO'A$ is double the angle QOA or θ. Hence it is evident that the limaçon may be generated by the epicyclic method of Art. 285. The radius $O'Q$ here revolves with double the angular rate of QP. Thus the equations of the curve, as referred to the origin O', may be written in the form

$$\left.\begin{aligned} x &= b \cos\theta + a \cos 2\theta, \\ y &= b \sin\theta + a \sin 2\theta, \end{aligned}\right\}$$

in which, comparing with equations (6), Art. 285,

$$R = b, \qquad c = a, \qquad m = 2.$$

Since m is positive, the curve is an epitrochoid, and denoting the values of its constants when written in the form (3), Art. 283, by a', b' and c', they are, by equations (1), Art. 287,

$$a' = \tfrac{1}{2}b, \qquad b' = \tfrac{1}{2}b, \qquad c' = a.$$

Thus the limaçon is an epitrochoid in which the fixed and rolling circles are equal.

310. In accordance with Art. 288, the limaçon may also be generated as a hypotrochoid, the constants for this mode being, by equations (2), Art. 287,

$$a'' = -a, \qquad b'' = 2a, \qquad c'' = b.$$

Thus the fixed circle is, in this case, identical with the protracted circle drawn in Fig. 55, and the rolling circle has a radius equal to the diameter of that circle. The initial position of the point of contact is at O, and the centre of the rolling circle lies always on the circumference of the fixed one; for example, it is at Q in Fig. 55 when the generating point is at P.

The limaçon in which $b > 2a$ does not pass through the pole O, and therefore forms a single oval as in Fig. 56.

The Dygogram.

311. The limaçon occurs in the graphic representation of magnetic forces introduced by Archibald Smith, Esq.,* in

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* *Transactions of the Institution of Naval Architects*, vol iii, p. 70.

1862, into the Theory of the Deviation of the Compass in iron ships. By the principles of Mechanics, the several forces are represented by lines having proper directions and magnitudes, and their joint effect is obtained by laying them off successively, end from end, beginning at a fixed point. The forces which vary in direction relatively to the meridian for different positions of the ship's head are two in number, known respectively as the *semicircular* and the *quadrantal* deviating forces. They are constant in magnitude; and, as the ship swings completely round, the former makes one revolution in direction and the latter two. Hence, when laid off from O' (the end of a line representing the constant force), they take just such positions as $O'Q$ and QP in Figs. 55 and 56, which revolve one at twice the angular rate of the other, and the final point P describes a limaçon as explained in Art. 309.

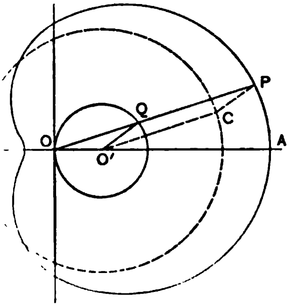

FIG. 56.

The limaçon thus used, with points corresponding to different headings of the ship marked upon its perimeter, is called *the Dygogram,** and serves to determine the direction of the total magnetic force for every position of the ship's head.

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* A contraction of dynamo—gonio—gram from δύναμις, force, and γωνία, angle. Mr. Smith in his diagram laid off from the fixed point the slowly rotating line representing the semicircular force (which is usually the larger, as $O'C$ in Fig. 56). The locus of its extremity is the dotted circle. From points of this circle the lines representing the quadrantal force were then drawn at inclinations in each case double that of $O'C$, as in the construction of the epicycle. The opposite order of construction, as in *Diehl's Compensation of the Compass, p.* 31, corresponds more directly to the definition of the curve as a limaçon.

The Cardioid.

312. When $b = 2a$, the limaçon becomes a cusped curve, which, by Art. 309, is the epicycloid formed when the fixed and the rolling circles are equal. In Fig. 57 the cardioid is drawn with its cusp on the right. Denoting the *diameter* of the fixed circle by a, the polar equation of the curve in this position is

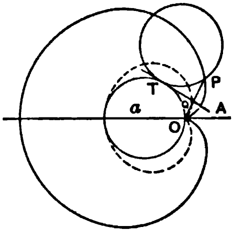

Fig. 57.

$$r = a(1 - \cos\theta) = 2a \sin^2 \tfrac{1}{2}\theta. \quad . \quad (1)$$

If the tangent common to the fixed and the rolling circle at their point of contact be drawn, the equality of the arcs OT and PT shows that the radius vector OP is bisected at right angles by the tangent at Q. It follows that the locus of Q is

$$r = a \sin^2 \tfrac{1}{2}\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a cardioid of one-half the linear dimension of the locus of P. The locus of the foot of the perpendicular from a fixed point upon a tangent to a given curve is called a *pedal* of the given curve. Hence it follows that the cardioid (2), represented by the dotted line in the figure, is *the pedal of the circle* whose diameter is a, *with respect to a point on its circumference.*

Transformation to Rectangular Coordinates.

313. The equations of transformation from rectangular to polar coordinates, the origin being the pole and the axis of x the initial line, are

$$x = r \cos\theta, \qquad y = r \sin\theta.$$

In the reverse transformation the trigonometric functions are expressed in terms of $\sin\theta$ and $\cos\theta$. Then, substituting y/r and x/r respectively for these, and clearing of fractions, r is finally eliminated by means of the relation

$$r^2 = x^2 + y^2.$$

For example, in the case of the limaçon, equation (2), Art. 308, becomes

$$r = 2a\frac{x}{r} + b;$$

whence

$$x^2 + y^2 = 2ax + b\sqrt{(x^2 + y^2)},$$

and clearing of radicals, we have

$$(x^2 + y^2)^2 - 4ax(x^2 + y^2) + (4a^2 - b^2)x^2 - b^2y^2 = 0$$

for the rectangular equation.

The equation indicates a double point at the origin; but, when $b > 4a$, the tangents at the origin become imaginary and the origin appears as an isolated point, the curve then having the form given in Fig. 56.

The rectangular equation of the cardioid, equation (1), Art. 313, is

$$(x^2 + y^2)^2 + 2a(x^2 + y^2) - a^2y^2 = 0.$$

Infinite Values of r.

314. When r becomes infinite for a finite value of θ, the curve $r = f(\theta)$ has a point at infinity in the direction indicated by this value of θ. Compare Art. 261, in which the direction ratio m is the value of the ratio $\frac{y}{x}$ or tan θ, when x, y and r become infinite.

The infinite branch of the curve will have an asymptote, if the perpendicular upon the tangent has a finite limit when the point of tangency recedes to infinity. It is obvious from Fig. 52, p. 293, that when P is at infinity the subtangent OT coincides with p. Hence if θ_1 is the value of θ which makes r infinite, the perpendicular upon the asymptote will be $\left. r^2 \frac{d\theta}{dr} \right]_{\theta_1}$ or $\left. \frac{d\theta}{du} \right]_{\theta_1}$, the former being laid off in the direction $\theta_1 - 90°$ when positive, and the latter in the direction $\theta + 90°$ when positive.

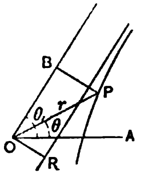

FIG. 58.

315. The expression deduced below is sometimes more convenient. In Fig. 58 the line OB is drawn through the pole in the direction θ_1. Then, dropping the perpendicular PB from a point P of the curve upon this line, we have, from the triangle OPB,

$$PB = r \sin (\theta_1 - \theta).$$

Now, if the curve has an asymptote parallel to OB, it is plain that, as θ approaches θ_1, the limiting value of PB will be equal to OR, the perpendicular from the pole upon the asymptote. Hence, if the curve has an asymptote in the direction θ_1, the expression

$$OR = [r \sin (\theta_1 - \theta)]_{\theta_1},$$

which takes the form $\infty \cdot 0$, will have a finite value, and this value will determine the distance of the asymptote from the pole. Fig. 58 shows that when the above expression is positive OR is to be laid off in the direction $\theta_1 - 90°$.

If, upon evaluation, the expression for OR is found to be infinite, we infer that the infinite branch of the curve is parabolic.

316. In the special cases, when $\theta_1 = 0$ and when $\theta_1 = 90°$, the expression for PB becomes $-r \sin \theta$ and $r \cos \theta$, respectively. In rectangular coordinates (the axis of x being the initial line, and origin the pole), these are $-y$ and x respectively. In fact it is obvious that in these cases, namely when the asymptote is parallel to one of the axes, the perpendicular from the origin is either the value of y when x is infinite, or the value of x when y is infinite.

For example, in the polar equation

$$r = a (\sec \theta + \tan \theta),$$

r is infinite when $\theta = 90°$; hence we take

$$x = r \cos \theta = a(1 + \sin \theta).$$

The value of this when $\theta = 90°$ is $2a$; there is therefore an asymptote perpendicular to the initial line at the distance $2a$ from the pole. In other words $x = 2a$ is the rectangular equation of the asymptote.

The curve in this illustration is the Strophoid, Fig. 40, p. 257, referred to the vertex of the loop as pole.

Points of Inflexion.

317. When, as in Fig. 52, the curve lies between the tangent and the pole, it is obvious that r and p will increase and decrease together; that is, $\dfrac{dp}{dr}$ will be positive. When, on the other hand, the curve lies on the other side of the tangent,

$\frac{dp}{dr}$ will be negative. Hence at a point of inflexion $\frac{dp}{dr}$ must change sign. It follows that, except at the pole, a point of inflexion can occur only where $\frac{dp}{dr} = 0$.

318. This criterion can be put in a form more convenient in application as follows: Taking the derivative of equation (2), Art. 300, with respect to u,

$$-\frac{2}{p^3}\frac{dp}{du} = 2u + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\frac{d\theta}{du};$$

hence

$$\left(u + \frac{d^2u}{d\theta^2}\right) = -\frac{1}{p^3}\frac{dp}{du},$$

and since

$$du = -\frac{dr}{r^2},$$

this may be written in the form

$$u + \frac{d^2u}{d\theta^2} = \frac{r^2}{p^3}\cdot\frac{dp}{dr}.$$

Now, since p is always positive, it follows that the sign of $\frac{dp}{dr}$ is the same as that of

$$u + \frac{d^2u}{d\theta^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

hence *at a point of inflexion this expression must change sign.*

Spirals.

319. When $f(\theta)$ in the polar equation $r = f(\theta)$ is not a periodic function, the curve is not completed when θ passes over a limited range of values, and it becomes necessary to consider all values of θ from $+\infty$ to $-\infty$. Curves of this kind are called *Spirals*. Successive revolutions of the radius vector give an unlimited number of successive portions or *whorls* of the spiral.

Let us first suppose that the infinite value of θ gives a finite value of r. In this case there will be an *asymptotic circle*; that is to say a circle to whose circumference the successive whorls of the spiral approach indefinitely.

320. As an illustration, let us take the equation

$$r = \frac{a\theta^2}{\theta^2 - 1}. \quad . \quad . \quad . \quad . \quad (1)$$

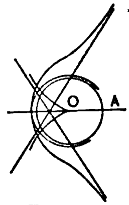

FIG. 59.

Equal positive and negative values of θ give the same value of r; hence the curve is symmetrical to the initial line. When θ increases without limit, r approaches the limiting value a, the point (r, θ) describes an infinite number of whorls approaching without limit to the asymptotic circle $r = a$ drawn in Fig. 59. When θ varies from 0 and 1, r is negative and varies from 0 to infinity, (r, θ) describing the branch in the third quadrant. Evaluating the expression for the perpendicular upon the asymptote, Art. 315, we have

$$[r \sin(\theta_1 - \theta)]_{\theta_1} = \left[\frac{a\theta^2}{\theta + 1} \cdot \frac{\sin(1 - \theta)}{\theta - 1}\right]_1 = -\tfrac{1}{2}a.$$

The angle $\theta = 1$ is the radian corresponding to 57° 18′, nearly, and, since the expression for the perpendicular on the asymptote is negative, its direction is $\theta_1 + 90° = 147° \ 18'$; consequently, the position of the asymptote is that given in Fig. 59. When $\theta > 1$, r becomes positive and rapidly decreases to approach the limiting value a.

321. It is obvious that the branch thus described contains a point of inflexion. To determine its position we employ the criterion of Art. 318. Thus, equation (1) is equivalent to

$$u = \frac{1}{a}\left(1 - \theta^{-2}\right);$$

whence $\dfrac{du}{d\theta} = \dfrac{2}{a}\theta^{-3}$, and $\dfrac{d^2u}{d\theta^2} = -\dfrac{6}{a}\theta^{-4}$;

therefore $$u + \frac{d^2u}{d\theta^2} = \frac{1}{a}\left(1 - \theta^{-2} - 6\theta^{-4}\right)$$

$$= \frac{\theta^4 - \theta^2 - 6}{a\theta^4}.$$

Putting this expression equal to zero, the real roots are

$$\theta = \pm \sqrt{3},$$

and it is evident that, as θ passes through either of these values, the expression $u + \dfrac{d^2u}{d\theta^2}$ changes sign. Hence the points of inflexion are determined by

$$\theta = \pm \sqrt{3}. \quad \text{and} \quad r = \frac{3a}{2}.$$

The angle $\theta = \sqrt{3}$ corresponds to nearly 100°.

The Spiral of Archimedes.

322. When the infinite value of θ makes r infinite, the spiral has an infinite number of whorls of increasing magnitude. An example is furnished by the simplest of all spirals, that of *Archimedes*, in which the radius vector is proportional to the angular coordinate, so that the equation is

$$r = a\theta. \quad . \quad . \quad . \quad . \quad (1)$$

The distance between successive whorls is constant and equal to $2\pi a$. Since $r = 0$ when $\theta = 0$, the curve touches the initial lines at the pole, as in Fig. 60.

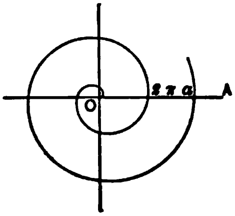

FIG. 60.

The equation $r = a\theta + b$ represents the same curve in a different position, for we may write it in the form $r = a(\theta + \theta_0)$, showing that the curve differs only in the direction with which it reaches the pole. This spiral is, in fact, the only curve which is identical with its protractions.

Negative values of r in equation (1) give a similar spiral symmetrically situated to a perpendicular to the initial line.

The Reciprocal Spiral.

323. When the infinite value of r makes $r = 0$, the spiral has an infinite number of whorls approaching the pole but never reaching it. The *Reciprocal Spiral* whose equation is

$$r = \frac{a}{\theta}$$

furnishes an example. See Fig. 61. The value of a in the diagram is that of r when θ is the radian about 57°.3. When $\theta = 0$, $r = \infty$. Hence there is a point at infinity in the direction of the initial

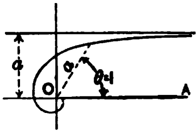

FIG. 61.

line; and, proceeding as in Art. 316, we have

$$y = r \sin\theta = \frac{a \sin\theta}{\theta}.$$

Evaluating this fraction for $\theta = 0$, we have $y = a$; hence the line

$$y = a$$

is an asymptote.

The Logarithmic or Equiangular Spiral.

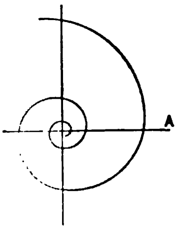

FIG. 62.

324. The spiral whose equation is

$$r = ae^{n\theta} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\log r = \log a + n\theta \quad . \quad . \quad . \quad (2)$$

is called the *Logarithmic Spiral.* Supposing n to be positive, $r = \infty$ when $\theta = +\infty$, and $r = 0$ when $\theta = -\infty$. The curve therefore consists of an infinite number of increasing whorls when θ is positive, and it approaches the pole with an infinite number of decreasing whorls when θ is negative. See Fig. 62.

Substituting in equation (4), Art. 297, we find for this curve

$$\cot\psi = n;$$

therefore the curve makes a constant angle with its radius vector. For this reason, it is also called the *Equiangular Spiral.*[*]

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[*] From the property of the stereographic projection of the sphere, that angles are unchanged in magnitude, it follows that the stereographic projection upon the equator of the Loxodromic curve (which cuts the meridians at a constant angle) is an equiangular spiral.

It is readily seen that successive whorls cut any radius vector in points whose distances from the pole are in geometrical progression. Again, the equation $r = be^{n\theta}$ represents the same spiral, only so turned about the pole that the radius vector whose length is b coincides with the initial line. Thus the logarithmic spiral is identical with all its similar curves.

Examples XXVII.

1. Prove that, in the case of the lemniscate $r^2 = a^2 \cos 2\theta$,

$$\psi = 2\theta + \tfrac{1}{2}\pi, \quad \frac{ds}{d\theta} = \frac{a^2}{r}, \quad \text{and} \quad p = \frac{r^3}{a^2}.$$

2. Find the value of p in the case of the curve $r^n = a^n \sin n\theta$.

$$p = a\,(\sin n\theta)^{1+\frac{1}{n}}.$$

3. In the case of the parabola referred to the focus

$$r = \frac{2a}{1 + \cos\theta},$$

prove that $p^2 = ar$.

4. In the case of the equilateral hyperbola $r^2 \cos 2\theta = a^2$, prove that $p = \dfrac{a^2}{r}$.

5. In the case of the ellipse $r = \dfrac{a(1 - e^2)}{1 - e\cos\theta}$, the pole being at the focus, determine p.

$$p = \frac{a(1 - e^2)}{\sqrt{(1 - 2e\cos\theta + e^2)}}.$$

6. In the case of the cardioid $r = a(1 - \cos\theta)$, prove that $\psi = \tfrac{1}{2}\theta$, and that $r^3 = 2ap^2$.

7. Trace the spiral $r^2 = \dfrac{a^2}{\theta}$, which is known as *the Lituus*, finding the asymptote and the point of inflexion.

8. Find the polar subtangent of the spiral $r(e_\theta + e^{-\theta}) = a$.

$$-\frac{a}{e_\theta - e^{-\theta}}.$$

9. Find the subtangent and the subnormal of the spiral of Archimedes, and prove that $\frac{ds}{dr} = \frac{\sqrt{(a^2 + r^2)}}{a}$.

10. Determine the asymptotes of the hyperbola from its polar equation referred to the focus, namely,

$$r = \frac{a(e^2 - 1)}{1 - e\cos\theta}$$

$$\theta_1 = \pm \sec^{-1} e, \quad p = \mp a\sqrt{(e^2 - 1)}.$$

11. Prove that the condition which determines points of inflexion in polar coordinates, namely, that $u + \frac{d^2u}{d\theta^2}$ shall change sign, is equivalent to the condition that $r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}$ shall change sign.

12. Show that the curve $r\theta \sin\theta = a$ has a point of inflexion at which $r = \frac{2a}{\pi}$.

13. Show that the curve $r\theta^m = a$ has points of inflexion determined by $\theta = \sqrt{[m(1 - m)]}$.

14. Show that, if the curve $r = \frac{f(\theta)}{F(\theta)}$ has an asymptote whose inclination to the initial line is θ_1, the perpendicular on it will be

$$-\frac{f(\theta_1)}{F'(\theta_1)},$$

except when $f'(\theta_1)$ is infinite.

15. Show that the curve $r(2\theta - 1) = 2a\theta$ has an asymptote determined by $\theta = \frac{1}{2}$, $p = -\frac{1}{2}a$, and a point of inflexion determined by the real root of the equation $2\theta^3 - \theta^2 - 2 = 0$.

16. The *Conchoid of Nicomedes* is defined as the protraction of a straight line, so that its polar equation is

$$r = a \sec \theta \pm b.$$

(Compare Ex. XXIV. 20.) Show from this equation that the maximum ordinate corresponds to $\cos^3\theta = \frac{a}{b}$, and the points of inflexion to a root of

$$b^2 \cos^4\theta + ab \cos^3\theta - 4a^2 \cos^2\theta - 2ab \cos \theta + 4a^2 = 0.$$

17. Trace the curve $r = a(2 \sin \theta - 3 \sin^3\theta)$.

18. Trace the curve $(x^2 + y^2)^3 - 4a^2x^2y^2 = 0$, first converting to polar coordinates.

19. Trace the curve $r^3 = a^3 \sin 3\theta$.

20. Trace the curve $r = a \cos^3 \frac{1}{3}\theta$, finding maximum ordinates and abscissæ.

21. Trace the curve $r = 2 + \sin \frac{1}{3}\theta$, finding three double points.

22. Trace the curve $r \cos \theta = a \cos 2\theta$, finding an asymptote.

23. Trace the curve $r \cos 2\theta = a$.

24. Trace the curve $r^2 = a^2 \cos 2\theta$.

25. Trace the curve $r = a(\cos \theta + \cos 2\theta)$, showing that there is a cusp at the origin and two double points.

26. Trace the curve $r^2 \sin \theta = a^2 \cos 2\theta$.

27. Trace the curve $r^2 \cos \theta = a^2 \sin 3\theta$.

28. Trace the curve $r = \dfrac{a\theta}{\theta + \sin \theta}$.

29. The locus of the points the product of whose distances from two fixed points is constant is known as *the Cassinian Oval.* Show that the polar equation of the Cassinian, when the fixed points are $(\pm a, 0)$ and the constant product c^2, is

$$r^4 - 2a^2r^2 \cos 2\theta + a^4 - c^4 = 0.$$

Show that the curve becomes a lemniscate when $c = a$, and consists of a single oval surrounding the lemniscate or two ovals within its loops, according as $c>a$ or $c<a$. In the latter case, determine the values of θ and r when r is tangent to the curve.

$$\sin 2\theta = \pm\frac{c^2}{a^2}; \quad r = \sqrt[4]{(a^4 - c^4)}.$$

30. Show that the rectangular equation of the Cassinian is

$$(x^2 + y^2)^2 + 2a^2(y^2 - x^2) + a^4 - c^4 = 0;$$

thence show that the curve has maximum ordinates at points where $r = a$, and that double tangents and points of inflexion exist when $2a > c > a$.

XXVIII.

The Measure of Curvature.

325. Regarding a line as the path of a moving point, its *curvature* is measured by the rate at which the direction of the motion changes as the point moves uniformly along its path. Thus the curvature of a straight line is zero, because there is no change of direction. The curvature of a circle is the same for all its points, because if a point P moves uniformly in the circumference its direction (being always perpendicular to that of the uniformly rotating radius OP) changes at a uniform rate.

For any curve other than the circle, the curvature varies from point to point. Denoting, as usual, the arc described by the moving point by ds, and by ϕ the inclination of the curve at P to the axis of x, $\frac{d\phi}{dt}$ is the rate of change of direction, and $\frac{ds}{dt}$ is the rate of the point's motion. Hence the relative rate $\frac{d\phi}{ds}$ is the measure of the varying curvature.

The Radius and Circle of Curvature.

326. When the point P moves with a constant linear rate in the circle whose centre is O and whose radius is a, the arc described in the time dt is $ds = a d\phi$, because OP revolves with the angular rate $\frac{d\phi}{dt}$, so that ds subtends the differential angle $d\phi$ at the centre O. Therefore

$$\frac{d\phi}{ds} = \frac{1}{a};$$

that is to say, *the measure of the curvature of a circle is the reciprocal of its radius.*

When the curvature is variable, the reciprocal of the measure of curvature at any point is called *the radius of curvature* for that point, and is denoted by ρ. Thus

$$\rho = \frac{ds}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the radius of a circle of which the curvature is the same as that of the given curve at the given point. In Fig. 63, let PC, equal to the value of ρ for the point P of the curve AP, be laid off upon the normal on the concave side of the curve. Then the circle with centre at C, and radius equal to ρ, not only touches the given curve at P, but has the same curvature. This circle, represented by the dotted line in Fig. 63, is called *the circle of curvature* for the point P. If s is measured from the fixed point A, so that the direction AP is posi-

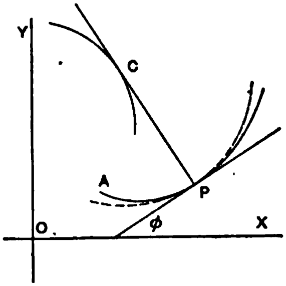

FIG. 63.

tive, the figure represents the case in which ϕ increases with s, and therefore ρ is positive. Hence we infer that, when ρ is positive, it should be laid off from P in the direction $\phi + 90°$. The point C so found is called *the centre of curvature.*

327. In Fig. 19, p. 97, a group of curves is represented passing through a common point in a common direction. Of the two curves AB and $A'B'$ lying above the tangent line, $A'B'$, which has the least curvature, (and therefore the greatest radius of curvature,) lies between the curve AB and the tangent, on each side of the point of tangency. This will necessarily be the case, at least in the immediate neighborhood of the point of contact, when the radii of curvature differ in value. Thus, if a number of circles were drawn touching the given curve at P, in Fig. 63, any one with radius greater than ρ would lie on the convex side of the curve in the immediate neighborhood of P, and any one with radius less than ρ would lie on the concave side.

Now, except at points where the radius of curvature has either a maximum or a minimum value, it will be increasing or decreasing in value as the moving point passes through the given position. For instance, in Fig. 63 it is represented as increasing as we pass through P in the direction AP. Accordingly, the curve lies on the concave side of the circle on the side of P toward A, and on the convex side of the circle beyond P. Thus the circle of curvature generally crosses the curve, as well as touches it, at the point of contact.

328. When the rectangular coordinates of a curve are given in terms of a third variable, the value of ρ is readily expressed in terms of this variable.

For example, in the case of the cycloid, equations (1), Art. 278, the values of ϕ and of ds are given in equations (1) and (2), Art. 279. Substituting in the expression for ρ,

$$\rho = \frac{ds}{d\phi} = -4a \sin \tfrac{1}{2}\psi.$$

This result shows that ρ is in value double of the chord PR in Fig. 43.

In particular, putting $\psi = 0$, we see that ρ vanishes at the cusp, and putting $\psi = \pi$, ρ has for the vertex O' its maximum value $4a$. In accordance with the preceding article, the circle of curvature for the vertex will lie on the upper or concave side of the curve on each side of the vertex. A slightly smaller circle touching the curve at the vertex would lie below the curve in the immediate neighborhood of the vertex and would cut it in points on each side.

The Radius of Curvature where the Curve is Parallel to one of the Axes.

329. When $\phi = 0$, that is, at points where the tangent to the curve is parallel to the axis of x, we have $dy = 0$ and therefore $ds = dx$, and also $d \tan \phi = d\phi$. It follows that at these points

$$\frac{d\phi}{ds} = \frac{d \tan \phi}{dx} = \frac{d^2y}{dx^2}.$$

Denoting the radius of curvature at this particular point by ρ_0, we have then

$$\rho_0 = \frac{1}{\dfrac{d^2y}{dx^2}}. \qquad \qquad (1)$$

In like manner, at a point where the curve is parallel to the axis of y, the measure of curvature is $\dfrac{d^2x}{dy^2}$. Hence for a "vertical" point of the curve, we have

$$\rho_1 = \frac{1}{\dfrac{d^2x}{dy^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The values of the special derivatives involved in these formulæ are very readily found in the case of any algebraic curve. For we have seen, in Art. 168, that when (a, b) is a point of the given curve,

$$\left.\frac{y-b}{x-a}\right]_{a,b} = \left.\frac{dy}{dx}\right]_{a,b}.$$

But, if $\phi = 0$ at the point (a, b), the value of this expression is zero, so that $\dfrac{y-b}{(x-a)^2}$ takes the indeterminate form $\dfrac{0}{0}$. Evaluating this last expression, we have

$$\left.\frac{y-b}{(x-a)^2}\right]_{a,b} = \left.\frac{\dfrac{dy}{dx}}{2(x-a)}\right]_{a,b} = \frac{0}{0} = \frac{1}{2}\left.\frac{d^2y}{dx^2}\right]_{a,b}.$$

Therefore $\left.\dfrac{d^2y}{dx^2}\right]_{a,b} = 2\left.\dfrac{y-b}{(x-a)^2}\right]_{a,b}$, and substituting in equation (1),

$$\rho_0 = \left.\frac{(x-a)^2}{2(y-b)}\right]_{a,b}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

In like manner, we obtain for the vertical points of the curve

$$\rho_1 = \left.\frac{(y-b)^2}{2(x-a)}\right]_{a,b}. \quad . \quad . \quad . \quad . \qquad (4)$$

330. For example, in the case of the ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the curve is parallel to the axis of x at the extremity $(0, b)$ of the minor axis; whence, for this point, by equation (3),

$$\rho_0=\left.\frac{x^2}{2(y-b)}\right]_{0,b}.$$

Putting the equation in the form

$$x^2=\frac{a^2}{b^2}(y^2-b^2),$$

we derive

$$\rho_0=\left.\frac{x^2}{2(y-b)}\right]_{0,b}=\left.\frac{a^2(y+b)}{2b^2}\right]_{0,b}=\frac{a^2}{b}.$$

In like manner, the radius of curvature at the point $(a, 0)$ is found to be $\frac{b^2}{a}$.

The Locus of the Centre of Curvature, or Evolute.

331. A clear conception of the mode in which the curvature of a given curve varies is best obtained by a consideration of *the locus of the centre of the circle of curvature*. Suppose P in Fig. 63 to move along the curve, carrying with it a plane in which are drawn two fixed straight lines intersecting at right angles, and let these lines coincide at every instant with the tangent and normal at P to the given curve. Then the motion of this plane, as it slides over the fixed plane, consists of a rotation about P combined with the motion due to the linear motion of

P. Now the motion of this plane might equally well be defined as a rotation about any other point of it, combined with the linear motion of that point. Let us take, for this purpose, that point of the moving plane which, at the instant represented in the figure, is situated at C. This point has no motion at the instant; for, if each of the two component motions first mentioned became uniform, the motion of the plane would become simple rotation about C, P then describing the circle of curvature. C is therefore called *the instantaneous centre* of the motion of the plane.

It follows that the motion of the centre of curvature in the fixed plane is simply that due to the change in the value of ρ; in other words, C moves in the direction of the normal to the given curve, and at the rate $\dfrac{d\rho}{dt}$, which is its rate in the moving plane, as it moves along the line PC of that plane. The curve described by C is called *the evolute* of the given curve, and the motion of the plane may now be defined by the *rolling* of the line PC drawn in it upon the evolute drawn in the fixed plane.

332. As an illustration, the evolute of the ellipse is drawn in Fig. 64. The values of AC_1 and BC_2, the radii of curvature at the extremities of the axes, are $\dfrac{b^2}{a}$ and $\dfrac{a^2}{b}$ respectively, as found in Art. 330. As P moves from A to B in the ellipse, C moves from C_1 to C_2, describing an arc touching each axis. Similar arcs correspond to the other quadrants of the ellipse.

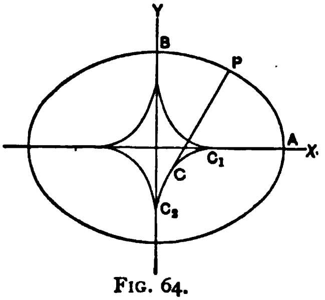

Fig. 64.

It will be noticed that cusps of the evolute correspond to maxima and minima values of ρ. If a solid piece having the arc C_1C_2 for its convex outline be made, we can suppose the quadrant BA of the ellipse to be described by the extremity of a string C_2B which is wound upon the arc C_2C_1.*

The Radius of Curvature in Rectangular Coordinates.

333. To express ρ in terms of derivatives with reference to x, we have

$$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}, \quad \text{and} \quad \phi = \tan^{-1}\frac{dy}{dx};$$

whence

$$\frac{d\phi}{dx} = \frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2},$$

therefore

$$\rho = \frac{\dfrac{ds}{dx}}{\dfrac{d\phi}{dx}} = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}. \quad . \quad . \quad . \quad . \quad (1)$$

In this expression, ρ has the sign of $\dfrac{d^2y}{dx^2}$, and therefore, by Art. 98, has the positive sign when the curve is concave as viewed from above, and *vice versa*. This will be found to agree with the rule for the direction of ρ given in Art. 326.

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* In the approximate construction of an arc of a curve by means of circular arcs, in mechanical drawing, the evolute of the arc to be drawn is practically assumed to be a polygon instead of a curve.

334. For example, to express the radius of curvature of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in terms of the abscissa, we have

$$y = \pm \frac{b}{a} \sqrt{(a^2 - x^2)}.$$

Differentiating,

$$\frac{dy}{dx} = \mp \frac{bx}{a\sqrt{(a^2 - x^2)}},$$

and

$$\frac{d^2y}{dx^2} = \mp \frac{ab}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Thus $1 + \left(\frac{dy}{dx}\right)^2 = \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)}$, and substituting in equation (1), Art. 333,

$$\rho = \mp \frac{[a^4 - (a^2 - b^2)x^2]^{\frac{3}{2}}}{a^4 b},$$

the upper sign corresponding to the upper or convex semi-ellipse, and the lower to the concave half. Supposing $b < a$, and putting $\sqrt{(a^2 - b^2)} = ae$, where e is the eccentricity of the ellipse, this becomes

$$\rho = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{a^2\sqrt{(1 - e^2)}}.$$

335. A more symmetrical expression for ρ is obtained by using derivatives with respect to a third variable, say t; we then have (compare equation (2), Art. 86)

$$\phi = \tan^{-1}\frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{whence} \quad \frac{d\phi}{dt} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Therefore

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}. \quad . \quad . \quad . \quad . \quad (2)$$

If in this expression t denotes time, it gives the value of ρ in terms of the component velocities and accelerations of P along the axes. By Art. 326, this value when positive is to be laid off on the left hand of the direction in which the point is moving.

When y is the independent variable, we may write

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is positive when the concave side of the curve is on the right, just as expression (1), Art. 333, is positive when the concave side is upward.

336. When s is taken as the independent variable, the numerator of expression (2) becomes unity, and the denominator, which is now the value of the measure of curvature, is

$$\frac{1}{\rho} = \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But this expression may be simplified by means of the relation

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right) = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and its derivative

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Eliminating successively $\frac{d^2x}{ds^2}$ and $\frac{d^2y}{ds^2}$ from equation (1), we derive

$$\rho = \frac{\frac{dx}{ds}}{\frac{d^2y}{ds^2}} = -\frac{\frac{dy}{ds}}{\frac{d^2x}{ds^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Again, eliminating the two first derivatives, we obtain

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

The Equation of the Evolute.

337. Denoting the coordinates of C in Fig. 63 by x' and y', we have, by projecting upon the axes the line $PC = \rho$, whose inclination to the axis of x is $\phi + 90$,

$$x' = x - \rho \sin \phi = x - \rho\frac{dy}{ds}, \quad . \quad . \quad . \qquad (1)$$

$$y' = y + \rho \cos \phi = y + \rho\frac{dx}{ds}. \quad . \quad . \quad . \qquad (2)$$

Hence, using equation (1), Art. 333,

$$x' = x - \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx}, \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

$$y' = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

If, in these equations, y and its derivatives are expressed in terms of x by means of the given equation, we have only to eliminate x to obtain the equation of the evolute.

For example, to find the evolute of the common parabola $y^2 = 4ax$, we have

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}, \qquad \text{whence} \qquad \frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}},$$

$$1+\left(\frac{dy}{dx}\right) = \frac{a+x}{x}, \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}.$$

Substituting in equations (3) and (4),

$$x' = 2a + 3x, \qquad y' = -\frac{2x^{\frac{3}{2}}}{a^{\frac{1}{2}}},$$

and eliminating x, we have

$$27ay'^2 = 4(x' - 2a)^3.$$

Therefore the evolute of the parabola is a semi-cubical parabola (Art. 250), having its cusp at the point $(2a, 0)$.

338. One or both of the equations (3) and (4) may be replaced by similar equations in which y is the independent variable, so that x and y are interchanged throughout. For example, using the results obtained in Art. 334, equation (3) gives for the ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x' = x - \frac{a^4x-(a^2-b^2)x^3}{a^4} = \frac{(a^2-b^2)x^3}{a^4}. \quad . \quad . \quad (2)$$

Since the equation of the ellipse is unchanged when we interchange x and y and at the same time a and b, we infer at once that the corresponding equation with y as independent variable would give

$$y' = \frac{(b^2-a^2)y^3}{a^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating both x and y between equations (1), (2) and (3), we obtain

$$(ax')^{\frac{2}{3}} + (by')^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}},$$

which is therefore the equation of the evolute drawn in Fig. 64.

339. Substituting $\dfrac{ds}{d\phi}$ for ρ in equations (1) and (2), Art. 337, we obtain the formulæ

$$x' = x - \frac{dy}{d\phi}, \qquad y' = y + \frac{dx}{d\phi}, \quad . \quad . \quad . \quad (1)$$

which are convenient when x and y are expressed in terms of a third variable. For example, in the case of the cycloid,

$$x = a(\psi - \sin \psi), \qquad y = a(1 - \cos \psi),$$

we have, as in Art. 279,

$$dx = a(1 - \cos \psi)d\psi, \qquad dy = a \sin \psi d\psi$$

and $\phi = 90 - \frac{1}{2}\psi$, whence $d\phi = -\frac{1}{2}d\psi$. Substituting in the formulæ, we find

$$x' = a(\psi + \sin \psi), \qquad y' = -a(1 - \cos \psi).$$

Comparing with the equations of the cycloid referred to its vertex, Art. 280, we see that the evolute is a similar cycloid below the axis of x, as in Fig. 65.

340. This result may also be derived geometrically from the fact, proved in Art. 328, that the radius of curvature is double of the chord PR of the generating circle. For it follows that Q, the centre of curvature, is a point of the equal circle RQR' situated below the axis of x. Hence the arc $R'Q$ is the supplement of PR and therefore equal to the distance $O'R'$, so that the circle RQR' rolling upon the lower horizontal line generates the locus of Q, which is thus a cycloid equal to the given one.

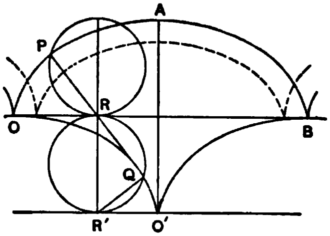

FIG. 65.

If solid pieces having the arcs of the lower cycloid as their convex outlines be employed, as in Art. 332, the upper cycloid may be described by the extremity of a string of length $O'A$, having the extremity O' fixed, and being wrapped upon the arc $O'O$ and $O'B$. It follows that the arc OO' is equal to $OA = 4a$; so that the whole length of a branch of the cycloid is eight times the radius of the generating circle.

If the figure be inverted it represents the method in which Huyghens proposed to make a particle oscillate in a cycloid, to illustrate his discovery of the *isochronism* of these vibrations.

The Radius of Curvature in Polar Coordinates.

341. When the curve is given by means of its polar equation we have, Fig. 51, Art. 297, $\phi = \psi + \theta$, whence

$$\rho = \frac{ds}{d\phi} = \frac{ds}{d\theta + d\psi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where

$$ds^2 = dr^2 + r^2 d\theta^2, \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and

$$\tan\psi = \frac{r\,d\theta}{dr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

In differentiating equation (3) to obtain an expression for $d\psi$, $d\theta$ may be regarded as constant; since the result is to be expressed in derivatives with reference to θ. Hence

$$\sec^2\psi\, d\psi = \frac{dr^2 - r\,d^2r}{dr^2} d\theta,$$

and, since $\sec\psi = \dfrac{ds}{dr}$,

$$d\psi = \frac{dr^2 - r\,d^2r}{ds^2} d\theta.$$

Hence

$$d\theta + d\psi = \frac{dr^2 + ds^2 - r\,d^2r}{ds^2} d\theta,$$

and, substituting in equation (1), we obtain

$$\rho = \frac{ds^3}{(dr^2 + ds^2 - r d^2r)d\theta};$$

therefore

$$\rho = \frac{(dr^2 + r^2 d\theta^2)^{\frac{3}{2}}}{(2dr^2 + r^2 d\theta^2 - r d^2r)d\theta},$$

or

$$\rho = \frac{\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}. \quad . \quad . \quad . \quad . \quad (4)$$

342. To obtain ρ in terms of u, Art. 298, we eliminate r from equation (4) thus:

$$r = \frac{1}{u}, \qquad \text{then} \qquad \frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta},$$

and

$$\frac{d^2r}{d\theta^2} = \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\theta^2}.$$

On substituting these values, we obtain

$$\rho = \frac{\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{u^3\left(u + \frac{d^2u}{d\theta^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It will be noticed that the denominator of this value of ρ contains the expression (1), Art. 318, which changes sign, and is therefore usually equal to zero, at a point of inflexion.

Relations between ρ, p and τ.

343. In Fig. 66, if we denote OR by p and PR by τ, we shall have

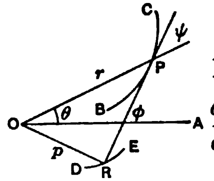

FIG. 66.

$$p = r\sin\psi, \quad \text{and} \quad \tau = r\cos\psi. \tag{1}$$

Now let P move along the curve at the rate $\frac{ds}{dt}$, then the tangent PR will rotate about P at the angular rate $\frac{d\phi}{dt}$, and OR will rotate about O at the same rate, because these lines are always at right angles to each other. The motion of the point R* may be resolved into two motions: one in the direction OR, and the other in the direction RP. Since the velocity of P in the direction OR is zero, the component of the velocity of R in this direction is $\tau\frac{d\phi}{dt}$, due to the rotation of PR about P, while the component in the direction RP is $p\frac{d\phi}{dt}$. The first of these components is the rate of p, since O is a fixed point; therefore

$$\frac{dp}{dt} = \tau\frac{d\phi}{dt}, \qquad \text{whence} \qquad \tau = \frac{dp}{d\phi}. \quad . \quad . \quad . \tag{2}$$

The rate of τ is the difference between the velocity of P in the direction RP, and the component velocity of R in the same direction; therefore

$$\frac{d\tau}{dt} = \frac{ds}{dt} - p\frac{d\phi}{dt}, \qquad \text{or} \qquad d\tau = ds - p\,d\phi. \quad . \tag{3}$$

---

* The locus of R is called the *pedal* (Art. 313) of the given curve from the origin O. The relation between p and ϕ gives the polar equation of the pedal, ϕ being the angular coordinate of the radius vector p when measured from an initial direction 90° behind OA, that is, downward in Fig. 66.

344. By comparing the expressions for τ in equations (1) and (2), and putting for $\cos\psi$ its value $\frac{dr}{ds}$, we obtain

$$\frac{dp}{d\phi} = r\frac{dr}{ds};$$

whence

$$\rho = \frac{ds}{d\phi} = \frac{rdr}{dp}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

An expression for ρ may also be derived from equation (3); thus,

$$\rho = \frac{ds}{d\phi} = p + \frac{d\tau}{d\phi}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or, by equation (2),

$$\rho = p + \frac{d^2p}{d\phi^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Intrinsic Equations.

345. The relation between s and ϕ for any curve is called its *intrinsic equation*, because it is independent of any geometrical elements exterior to the curve, except the direction $\phi = 0$, from which ϕ is measured, and the point on the curve from which s is measured. The intrinsic equation is usually put in the form $s = f(\phi)$; and, by preference, is so taken that s vanishes with ϕ, so that the initial direction (or $\phi = 0$) is that of the tangent to the curve at the initial point. Thus the intrinsic equation of the circle is written $s = a\phi$, and the addition of a constant only changes the point from which s is measured.

Differentiation of the intrinsic equation gives at once the value of ρ in terms of ϕ; on the other hand, the determination of s in terms of ϕ from the value of ρ, or of ds (as derived from the rectangular equation of the curve), requires in general the inverse process of Integration. Curves for which simple expressions for s exist are said to be *rectifiable*, and it is these curves which have simple intrinsic equations.

346. For example, in Fig. 65, we have seen that the arc OQ of the lower cycloid is equal to PQ or $2RQ$. Reckoning ϕ from the initial direction OB toward the right, ϕ is the angle QRB, and $RQ = 2a \sin \phi$; therefore

$$s = 4a \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the intrinsic equation of the cycloid referred to its vertex. This equation gives a maximum value for s, namely $4a$, when $\phi = \frac{1}{2}\pi$ at the point O'. Differentiation of equation (1) gives

$$ds = 4a \cos\phi \, d\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, if ϕ continues to increase beyond the value $\frac{1}{2}\pi$, ds changes sign, so that, while Q describes the arc $O'B$, s in equation (1) is the algebraic value of the arc with the new branch reckoned as negative. Thus s is again zero at the vertex B, and becomes $-4a$ at the next cusp.

347. As another illustration, let us find the intrinsic equation of the catenary (see note p. 217) of which the rectangular equation is

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = \tan \phi, \quad . \quad . \quad (2)$$

and

$$ds^2 = dx^2\left[1 + \frac{1}{4}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)^2\right] = \frac{dx^2}{4}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)^2.$$

Hence

$$\frac{ds}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and it is clear that, if we measure s from $(0, c)$, the point A in Fig. 67, p. 334, we must have

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

because this gives equation (3) by differentiation and makes $s = 0$ when $x = 0$. Comparing with equation (2), we have

$$s = c \tan \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for the intrinsic equation of the catenary.

The Intrinsic Equation and Radius of Curvature of the Evolute.

348. If s' and ϕ' denote the intrinsic coordinates of the evolute, we have seen in Arts. 332 and 326 that

$$ds' = d\rho, \quad \text{and} \quad \phi' = \phi + 90°, \quad \text{whence} \quad d\phi' = d\phi.$$

It follows that s' and ρ can differ only by a constant depending upon the initial point from which the arc is reckoned, so that the value of ρ in terms of ϕ gives the intrinsic equation of the evolute.

For example, the intrinsic equation of the catenary, equation (5) above, gives $\rho = c \sec^2\phi$. The minimum value c occurs at the vertex A, Fig. 67, giving the cusp A' of the evolute. If then we measure s' from the cusp, we shall have $s' = \rho - c = c \tan^2\phi$; and measuring ϕ' from the vertical direction, so that $\phi = \phi'$, we have

$$s' = c \tan^2\phi' \quad . \quad . \quad (6)$$

for the intrinsic equation of the evolute.

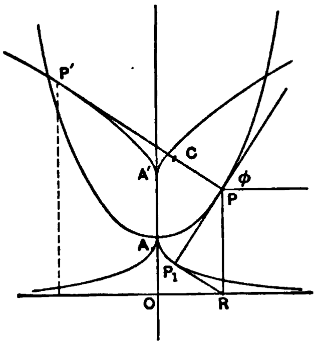

FIG. 67.

349. Since ρ', the radius of curvature of the evolute, is equal to $ds'/d\phi'$, we have

$$\rho = \frac{ds}{d\phi}, \qquad \rho' = \frac{d\rho}{d\phi} = \frac{d^2s}{d\phi^2}.$$

For example, for the evolute above, we find

$$\rho' = 2c \tan\phi \sec^2\phi.$$

In like manner, given the intrinsic equation of any curve, we can find the intrinsic equations and the radii of curvature of its successive evolutes.

Involutes and Parallel Curves.

350. When a tangent line rolls upon a given curve, any point on it describes a curve of which the given curve is the evolute. Such a curve is called *an involute* of the given curve, and other points on the tangent describe other involutes of the same curve. For example, the lower cycloid in Fig. 65 being given, the upper cycloid is the involute described by the point P, and the dotted line is the involute described by another point of the rolling tangent. The involutes of a given curve form a system of curves having common normals. With reference to one another, they are called *parallel curves* because the corresponding points of any two of them are at a constant distance measured along the normal.

351. Denoting this constant difference by c, if $s = f(\phi)$ is the intrinsic equation of a given curve, that of the parallel curve is $s = f(\phi) + c\phi$. For the radius of curvature of this curve is

$$\rho = \frac{ds}{d\phi} = f'(\phi) + c,$$

and that of the given curve is $f'(\phi)$. If (x, y) and (x_1, y_1) are corresponding points on the two curves referred to rectangular axes, the inclination of c to the axis of x is $\phi + 90$ (in Fig. 65, taking the cycloid as the given curve, c is negative); hence, projecting it upon the axes, we have

$$x_1 - x = -c \sin\phi = -c\frac{dy}{ds},$$

$$y_1 - y = c\cos\phi = c\frac{dx}{ds}.$$

Hence the equation of a parallel to a given curve is the result of eliminating x and y between these two equations and that of the given curve.

352. From the construction of the involute of a given curve it is obvious that a cusp occurs whenever the involute meets the curve at an ordinary point (that is at a point which is neither a cusp nor a point of inflexion). In Fig. 67, that involute which meets the catenary at the vertex A is drawn; so that, if ρ_1 denotes PP_1, the radius of curvature of the involute, we have $\rho_1 = s$. Equations (1) and (3), Art. 347, show that $y = c \sec \phi$. Hence if we join R, the foot of the ordinate, with P_1, and compare equation (5), we see that PP_1R is a right angle. Therefore RP_1 is tangent to the involute, and is equal to the constant c. It follows that this curve is the path of a heavy particle on a horizontal table when attached to the end of a string of length c, the other end of which is moved along the line OX. The curve is, for this reason, called *the Tractrix*.

353. A curve can have involutes which fail to meet it only in case there is a portion of the rolling tangent which the point of contact never reaches. For example, in Fig. 67, the point of contact P' moves along the rolling tangent PCP', but never passes beyond the fixed point C, at a distance c from P. Any point on the same side of C with P describes a curve without cusps. Points on the other side obviously describe curves with two cusps.

Again, in Fig. 65, the point of contact travels back and forth over a segment of the rolling line equal in length to one branch of the cycloid, and points beyond this segment describe looped curves without cusps. So also, in Fig. 64, the point of contact is confined to a segment equal in length to one branch of the cusped curve of which the ellipse is an involute. It is only in such a case as this, when the algebraic sum of the arcs of the given cusped curve is zero, that the involute can be a closed oval.

354. On the other hand, the involute of a closed oval will be a spiral having one cusp, and only one. For example, Fig. 68 represents the involute of the circle whose radius is a. In this particular case, all the involutes are evidently alike in shape; in other words, the involute of the circle is identical with its own parallels, a property shared only by the straight line.

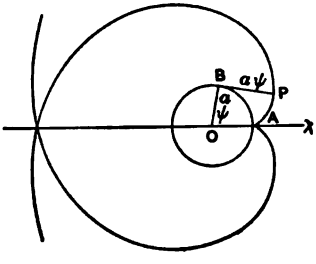

FIG. 68.

In Fig. 68, let O be taken as the origin of rectangular coordinates, and OA as the axis of x; then denoting AOB by ψ, $BP = a\psi$, and projecting OBP upon the axes, we have

$$\left.\begin{aligned} x &= a \cos \psi + a\psi \sin \psi, \\ y &= a \sin \psi - a\psi \cos \psi, \end{aligned}\right\}$$

for the rectangular equations of the involute of the circle.

The intrinsic equation of this curve is

$$s = \tfrac{1}{2}a\phi^2,$$

where ϕ for the point P is the same as ψ in Fig. 68; for this equation gives for the radius of curvature $\rho = a\phi$.

Examples XXVIII.

1. Find the radius of curvature of the parabola

$$\frac{\sqrt{x}}{\sqrt{a}} + \frac{\sqrt{y}}{\sqrt{b}} = 1$$

at the point where it touches the axis of x.

$$\rho_0 = \frac{2a^2}{b}.$$

2. Find the radius of curvature of the four-cusped hypocycloid

$$x = a \cos^3\psi, \qquad y = b \sin^3\psi.$$

$$\rho = -3a \sin \psi \cos \psi.$$

3. Find the radius of curvature of the *three-cusped hypocycloid*

$$x = a(2 \cos \psi + \cos 2\psi), \qquad y = a(2 \sin \psi - \sin 2\psi).$$

$$\rho = -8a \sin \tfrac{3}{2}\psi.$$

4. Find the radius of curvature of the curve

$$x = 2a \sin 2\psi \cos^2\psi, \qquad y = 2a \cos 2\psi \sin^2\psi.$$

$$\rho = 4a \cos 3\psi.$$

5. Find the radius of curvature of the *parabola* $y^2 = 4ax$.

$$\rho = \mp 2 \frac{(a + x)^{\frac{3}{2}}}{\sqrt{a}}.$$

6. Find the radius of curvature of the *catenary*

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

and show that its numerical value equals that of the normal at the same point.

$$\rho = \frac{y^2}{c}.$$

7. Find the radius of curvature of the *semi-cubical parabola*

$$ay^2 = x^3.$$

$$\rho = \frac{(4a + 9x)^{\frac{3}{2}} x^{\frac{1}{2}}}{6a}.$$

8. Find the radius of curvature of the *cissoid*

$$y = \frac{x^{\frac{3}{2}}}{(2a - x)^{\frac{1}{2}}}.$$

$$\rho = \frac{a\sqrt{x}(8a - 3x)^{\frac{3}{2}}}{3(2a - x)^2}.$$

9. Find the radius of curvature of the *parabola*

$$\sqrt{x} + \sqrt{y} = 2\sqrt{a}.$$

$$\rho = \frac{(x+y)^{\frac{3}{2}}}{\sqrt{a}}.$$

10. Find the radius of curvature of the *logarithmic curve*

$$y = ae^{\frac{x}{c}}.$$

$$\rho = \frac{[c^2+y^2]^{\frac{3}{2}}}{cy}.$$

11. Find the radius of curvature of the *hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$\rho = \mp\frac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}.$$

12. Find the radius of curvature of the *cubical parabola*

$$a^2y = x^3.$$

$$\rho = \frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^4x}.$$

13. Find the radius of curvature of the *prolate cycloid*

$$x = a\psi - b\sin\psi, \qquad y = a - b\cos\psi.$$

$$\rho = \frac{(a^2 + b^2 - 2ab\cos\psi)^{\frac{3}{2}}}{b(a\cos\psi - b)}.$$

14. Find the radius of curvature of the *rectangular hyperbola*

$$xy = m^2.$$

$$\rho = \frac{(x^2+y^2)^{\frac{3}{2}}}{2m^2}.$$

15. Show that when $n > 2$, the radius of curvature of the parabola of the nth degree is infinite at the origin, and is a minimum where

$$\frac{y}{x} = \frac{\sqrt{(n-2)}}{n\sqrt{(n-1)}}.$$

16. Given the curve $y^3 + x^3 + a(x^2 + y^2) = a^2y$. Find the value of ρ at the origin. $\rho_0 = \frac{a}{2}$.

17. Given the curve $ax^3 - 2b^2xy + cy^3 = x^4 + y^4$. Determine the values of ρ at the origin.

For the branch tangent to the axis of x, $\rho_0 = \frac{b^2}{a}$;

for the branch tangent to the axis of y, $\rho_1 = \frac{b^2}{c}$.

18. In the case of the *strophoid*, Fig. 40, p. 257, find the value of ρ at the vertex; also, after turning the axes through 45°, the value of ρ at the origin. $\frac{1}{4}a$; $\frac{1}{2}\sqrt{2} \cdot a$.

19. Given the curve $x^4 - ax^2y + axy^2 + \frac{1}{4}a^2y^2 = 0$. Determine the value of ρ_0. See Ex. XXV. 19. $\rho_0 = \frac{1}{4}a$.

20. Find the radius of curvature at the origin, the equation of the curve being

$$x^4 - \tfrac{5}{2}ax^2y - axy^2 + a^2y^2 = 0.$$

$\rho_0 = a$ and $\rho_0 = \frac{1}{4}a$.

21. Given the curve $x^5 - 4ay^4 + 2ax^3y + a^2xy^3 = 0$. Find the values of ρ_0. See Ex. XXV, 20.

$\rho_0 = -\frac{1}{8}a$, at the cusp;
$\rho = \frac{1}{8}a$, for the other branch.

22. Find the equation of the evolute of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

23. Prove that, in the case of the four-cusped hypocycloid

$$x = a\cos^3\psi, \qquad y = b\sin^3\psi,$$

the coordinates of the evolute satisfy

$$x' + y' = a(\cos\psi + \sin\psi)^3 \quad \text{and} \quad x - y = a(\cos\psi - \sin\psi)^3,$$

and thence deduce the rectangular equation of the evolute.

$$(x' + y')^{\frac{2}{3}} + (x' - y')^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

24. Given the equation of the *catenary*

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right);$$

prove that

$$y' = 2y, \qquad \text{and} \qquad x' = x - \frac{y}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),$$

and deduce the equation of the evolute. (See Fig. 67.)

$$\pm x' = a\log\frac{y' + (y'^2 - 4a^2)^{\frac{1}{2}}}{2a} - \frac{y'}{4a}(y'^2 - 4a^2)^{\frac{1}{2}}.$$

25. Find the equations of the evolute of the curve

$$x = c\sin 2\psi(1 + \cos 2\psi), \qquad y = c\cos 2\psi(1 - \cos 2\psi).$$

$$x' = 2c(-2\sin\psi\cos 3\psi + \sin 2\psi\cos^2\psi),$$
$$y' = 2c(2\cos\psi\cos 3\psi + \cos 2\psi\sin^2\psi).$$

26. Find the radius of curvature of the limaçon

$$r = a + b\cos\theta.$$

$$\rho = \frac{(a^2 + 2ab\cos\theta + b^2)^{\frac{3}{2}}}{a^2 + 3ab\cos\theta + 2b^2}.$$

27. Show that, for the lemniscate $r^2 = a^2\cos 2\theta$, $\rho = \frac{a^2}{3r}$; and that, for the equilateral hyperbola $r^2\cos 2\theta = a^2$, $\rho = -\frac{r^3}{a^2}$.

28. Find the radius of curvature of the conic referred to its focus

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}.$$

$$\rho = \frac{a(1 - e^2)(1 - 2e \cos \theta + e^2)^{\frac{3}{2}}}{(1 - e \cos \theta)^3}.$$

29. Find the radius of curvature of the *lituus*

$$r^2\theta = a^2.$$

$$\rho = \frac{r(4a^4 + r^4)^{\frac{3}{2}}}{2a^2(4a^4 - r^4)}.$$

30. Given $r^m = a^m \cos m\theta$, prove that $r^{m+1} = a^m p$, and thence by means of equation (4), Art. 344, prove that

$$\rho = \frac{r^2}{(m + 1)p}.$$

31. From the relation between p and r in the case of the parabola referred to its focus, Ex. XXVII. 3, prove by means of equation (4) Art. 344 the following construction for the radius of curvature of the parabola: Join P to the focus F and draw FN perpendicular to FP to meet the normal at P in N, then $\rho = 2PN$.

32. Show that the rectangular equations of the catenary in terms of a third variable are

$$x = c \log (\sec \phi + \tan \phi), \qquad y = c \sec \phi.$$

33. Show that the intrinsic equation of the cycloid as measured from a cusp is

$$s = 4a(1 - \cos \phi) = 8a \sin^2 \tfrac{1}{4}\phi.$$

34. Find the radius of curvature of the *epicycloid*

$$x = (a + b) \cos \psi - b \cos \frac{a + b}{b}\psi, \quad y = (a + b) \sin \psi - b \sin \frac{a + b}{b}\psi,$$

in which $\psi = 0$ corresponds to a cusp.

$$\rho = \frac{4b(a + b)}{a + 2b} \sin \frac{a\psi}{2b}.$$

35. Using the general equations of the epi- and hypocycloids

$$x = R\cos\phi + \frac{R}{m}\cos m\phi, \qquad y = R\sin\phi + \frac{R}{m}\sin m\phi,$$

Art. 285, in which $\psi = 0$ corresponds to a vertex, show that

$$\rho = \frac{4R}{m+1}\cos\tfrac{1}{2}(m-1)\psi.$$

36. Show that the intrinsic equation of the curve in Ex. 35 when $\phi = 0$ is the direction of the tangent at the vertex, is of the form

$$s = l\sin n\phi,$$

where $n = \dfrac{m-1}{m+1} = \dfrac{a}{a+2b}$, so that $n < 1$ for the epicycloid, and $n > 1$ for the hypocycloid. Hence show that the evolute is a similar curve of n times the linear dimensions of the given curve.

37. Denoting by r the number of cusps of an uncrossed epicycloid or hypocycloid, show that in the notation of Ex. 36 we have respectively

$$n = \frac{r}{r+2} \qquad \text{and} \qquad n = \frac{r}{r-2}.$$

38. Show that the intrinsic equation of the evolute of the four-cusped hypocycloid $s = l\sin 2\phi$, when measured from the cusp, is $s' = 4l\sin^2\phi$, and transform this equation to its own vertex and tangent.

39. Prove that the centre of curvature of the equiangular spiral, Art. 324, is the intersection of the normal and a perpendicular to r drawn through the pole; and thence that the evolute is a similar spiral which is identical with the given spiral turned forward through an angle $\dfrac{\pi}{2} - \dfrac{\log n}{n}$.

40. Show that the intrinsic equation of the equiangular spiral is

$$s = le^{n}\phi,$$

where l is the whole length of the curve measured from the pole given by $\phi = -\infty$ to the point where $\phi = 0$.

41. From the polar equation of the cardioid

$$r = 2a \sin^2 \tfrac{1}{2}\theta,$$

equation (1), Art. 312, derive the radius of curvature

$$\rho = \frac{4a}{3} \sin \tfrac{1}{2}\theta,$$

also the intrinsic equation referred to the cusp

$$s = 4a(1 - \cos \tfrac{1}{3}\phi),$$

and thence show that the evolute is a cardioid of one-third of the size of the given one.

42. Show that, in general: to an infinite branch corresponds a parabolic branch of the evolute; to a point of inflexion an asymptotic branch; and to a cusp a branch passing through the given point. What exceptional cases can occur?

XXIX.

Systems of Curves.

355. The constants to which arbitrary values may be assigned in the equation of a curve are called *parameters* of the curve, and the curves obtained by giving different values to the parameters are said to constitute a *system of curves*.

When a single parameter is considered and that admits of an infinite number of values (that is to say, of continuous variation), the system is called a *singly infinite* one. Denoting the variable parameter by α, the general equation of a singly infinite system is of the form

$$f(x, y, \alpha) = 0, \quad \dots\dots \quad (1)$$

in which different values of α distinguish different members of the system. If we suppose α to vary through all its possible values, the curve represented by equation ( ) sweeps over the whole or a portion of the plane. In fact, if we select any point P_1, and assume that its coordinates (x_1, y_1) satisfy equation (1), we have the equation $f(x_1, y_1, \alpha) = 0$, by which to determine the particular value or values of α for which the curve passes through the selected point.

356. If the equation is of the first degree in α, so that it can be put in the form

$$F_1(x, y) + \alpha F_2(x, y) = 0, \quad . \quad . \quad . \quad . \quad (1)$$

where F_1 and F_2 are one-valued* functions of x and y, we shall have, in general, for a selected position of P_1 a single value of α; so that one, and only one, member of the system of curves passes through the selected point. But, if the curves

$$F_1(x, y) = 0, \qquad F_2(x, y) = 0$$

intersect, each of the points of intersection will satisfy equation (1), *independently of* α, so that all the members of the system will pass through each of these points. It follows that the members of the system of curves intersect each other in no other points except these fixed ones, and that as α passes by continuous variation through all values from $-\infty$ to $+\infty$ the curve represented by equation (1) sweeps once over the whole plane.

357. A system of curves of this kind is called a *pencil of curves*. For example, the circles with centre on the axis of x

---

* Thus an equation containing radicals must be rationalized with respect to x and y, before its degree in α can be ascertained.

and cutting the axis of y in the point $(0, b)$ form such a system. For, if the centre of one of these circles is at $(\alpha, 0)$ its equation is $y^2 + (x - \alpha)^2 = b^2 + \alpha^2$, or

$$x^2 + y^2 - 2\alpha x - b^2 = 0. \quad . \quad . \quad . \quad . \quad (1)$$

The system of circles resulting from making α an arbitrary parameter all pass through the two points $(0, b)$ and $(0, -b)$, and may be described as the pencil of circles passing through these points, or having a common chord on the axis of y. Accordingly, the equation is of the first degree in α, and the fixed points of intersection are the intersections of

$$x^2 + y^2 - b^2 = 0 \qquad \text{and} \qquad x = 0.$$

If we change the sign of b^2 in equation (1) we still have a pencil of circles; the axis of y is in this case no longer a common chord, but in either case is called *the radical axis* of the pencil.

Envelopes.

358. In case the equation of the system is other than of the first degree in α, the curves of the system may intersect one another in other than fixed points. Let us suppose in the first place that

$$f(x, y, \alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is of the second degree in α. Then, when the coordinates (x, y) of a selected point P are substituted in equation (1), we have a quadratic equation for α. If the roots of this equation are real we find two values for α, determining two distinct curves of the system which intersect in P. Let P in Fig. 69 be such a point, and PA, PB the two curves of the system thus determined.

Suppose also that there are positions of P for which the values found for α are imaginary, so that no curves of the system pass through these points. Then, as P is moved from a position for which α is real to one for which α is imaginary, it will pass through a position for which the two values of α are equal. The locus of the points which give equal values of α will, in fact, constitute a *boundary line* separating a portion of the plane in which α is real from one in which it is imaginary.

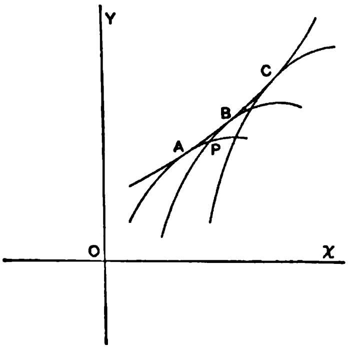

FIG. 69.

As P is moved up to the boundary line the two curves which there intersect come into coincidence. For this reason, the point on the boundary line is called *the ultimate intersection* of consecutive curves of the system.

In general, the ultimate intersections take place at ordinary points of the curves of the system, that is to say, at points which are not double points or cusps. When this is the case, as in Fig. 69, the boundary line touches the curves of the system, and is called *the envelope* of the system.

It follows that the envelope of a system of curves is the locus of the ultimate intersections or *points for which α has equal roots.*

Equations of the Second Degree in α.

359. When the equation of the system of curves is of the second degree in α, it may be written in the form

$$P\alpha^2 + Q\alpha + R = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P, Q and R are, in general, functions of x and y. The *discriminant* of this equation (or quantity which appears under the radical sign when it is solved for α) is $Q^2 - 4PR$; hence all points for which the values of α are equal must satisfy the equation

$$Q^2 - 4PR = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Consider now the locus of this equation in x and y. If in passing over any branch of this locus the discriminant changes sign, as will generally be the case, it will form a boundary between regions of the plane in which α is respectively real and imaginary, that is to say, it will be the locus of ultimate intersections and in general an envelope of the system. But if the discriminant does not change sign, the branch will not form a boundary, and will fail to give an envelope.

360. For example, the equation of the path of a projectile, neglecting the resistance of the air, is

$$y = x \tan \alpha - \frac{x^2}{4H \cos^2\alpha}, \quad . \quad . \quad . \quad . \quad (1)$$

representing a parabola in a vertical plane, the origin being the point of projection, and α the angle of projection. Supposing α to vary while H remains fixed, we have a system of parabolas in the same vertical plane, being the trajectories described *with a given initial velocity*. It is required to find the envelope of this system. Equation (1) is virtually one of the second degree in the arbitrary parameter; for, putting α for tan α, it may be written

$$4H(y - x\alpha) + (1 + \alpha^2)x^2 = 0,$$

or

$$x^2\alpha^2 - 4Hx\alpha + 4Hy + x^2 = 0.$$

Comparing with equation (1) of the preceding article $P = x^2$, $Q = -4Hx$ and $R = 4Hy + x^2$; hence equation (2) gives

$$x^2[4H^2 - (4Hy + x^2)] = 0$$

for the locus of points for which α has equal values. Here we reject the squared factor x^2 because, this factor being always positive, the discriminant does not change sign * when we cross its locus $x = 0$. The other factor gives the envelope

$$x^2 = 4H(H - y),$$

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* So in general we reject a factor of the discriminant which appears with an even exponent. In this example, the locus of the squared factor, $x^2 = 0$, is a member of the system, corresponding in fact to $\alpha = \infty$ (when the angle of projection is 90°).

A squared factor will also occur in the discriminant when all the curves of the system possess a double point or *node*. For, in that case, the two branches which pass through a selected point P (belonging, in general, to different curves of the system) will, when P is brought up to the *node-locus*, become branches of the same curve, and so correspond to a single value of α. Thus the node-locus is part of the locus for which two values of α have become equal; but the values remain real on both sides of the locus. In like manner, an isolated point or *acnode* gives rise to a squared factor in the discriminant and to a locus on which the values of α are equal, but the values are imaginary on each side of the locus.

When all the members of the system possess a cusp, the corresponding factor in the discriminant will occur as a factor of the third or higher odd-numbered degree, so that the *cusp-locus* does form a boundary between regions of α real and α imaginary, but it is easily distinguished from the envelope whose factor occurs in the first degree. In fact, the node locus and the envelope which would exist, if the members of the system were looped curves, may be regarded as coming into coincidence when they become cusped curves.

which represents a parabola with axis vertically downward and focus at the origin.

This envelope constitutes the boundary separating the points of the plane which can be hit by properly choosing α from those which cannot be reached. For points within it, there are two values of α, that is to say, two angles of elevation, which can be used. For a point upon it, there is but a single value of α.

General Method.

361. Denoting by $\Delta\alpha$ the difference between two values of α,

$$f(x, y, \alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

and

$$f(x, y, \alpha + \Delta\alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

are the equations of two members of the system of curves, such as those passing through A and B, Fig. 69, p. 347, which can be brought as near as we please to coincidence by diminishing $\Delta\alpha$. A point of intersection of these curves, such as P, Fig. 69, will satisfy both these equations, and therefore also their difference,

$$f(x, y, \alpha + \Delta\alpha) - f(x, y, \alpha) = 0. \quad . \quad . \quad . \qquad (3)$$

Regarding $f(x, y, \alpha)$ simply as a function of α, the first member of equation (3) is $\Delta f(x, y, \alpha)$, which vanishes identically when we put $\Delta\alpha = 0$ to obtain the ultimate intersection. But, if we divide by $\Delta\alpha$, the ratio $\Delta f/\Delta\alpha$ approaches a limiting value when Δx is diminished; for, by Art. 40, we have

$$\left.\frac{\Delta f(x, y, \alpha)}{\Delta\alpha}\right]_{\Delta\alpha = 0} = \frac{df(x, y, \alpha)}{d\alpha}.$$

Hence equation (3) becomes at the limit

$$\frac{df(x, y, \alpha)}{d\alpha} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and this equation is satisfied by that ultimate intersection which lies upon the curve (1), in other words, by the point of tangency of the curve (1) and the envelope. It follows that, if we eliminate α between equations (1) and (4), we shall have an equation satisfied by all the points of ultimate intersection. In other words, the equation of the envelope is the result of eliminating α between the equations

$$\left.\begin{aligned} f(x, y, \alpha) &= 0, \\ f'(x, y, \alpha) &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where f' stands for the derivative of f with respect to α. The result is called the discriminant of equation (1) with respect to α.

362. For example, let us find the envelope of the system of ellipses whose axes are fixed in position, and whose semi-axes have a constant sum. Denoting the constant sum by c, the equation of the varying ellipse is

$$\frac{x^2}{\alpha^2} + \frac{y^2}{(c - \alpha)^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Taking the derivative with respect to α,

$$-\frac{2x^2}{\alpha^3} + \frac{2y^2}{(c - \alpha)^3} = 0; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence

$$\alpha = c\frac{x^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}} \qquad \text{and} \qquad c - \alpha = c\,\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}};$$

and, substituting in equation (1), we find

$$\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^3=c^2.$$

Therefore the envelope is the four-cusped hypocycloid, **Art.** 292.

Two Variable Parameters.

363. When the given equation involves two parameters, α and β, it represents a doubly-infinite system of curves, unless there is a given relation between the parameters (like that between the semi-axes of the ellipse in the preceding example), so that one of them can be eliminated. Instead of eliminating one of the parameters at once, it is, however, sometimes preferable to proceed in the manner illustrated by the following example:

The centre of a circle which passes through the origin moves upon the equilateral hyperbola

$$x^2-y^2=a^2;$$

required the envelope. Taking for the two parameters the coordinates of the centre, the equation of the circle is

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2,$$

or

$$x^2+y^2-2\alpha x-2\beta y=0, \quad . \quad . \quad . \quad . \quad (1)$$

and the relation between the parameters is

$$\alpha^2-\beta^2=a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The derivative equation, to be combined with these, is now found by eliminating the ratio $d\beta : d\alpha$ from the differential equations. Thus, from equations (1) and (2), we have

$$xd\alpha = -yd\beta \qquad \text{and} \qquad \alpha d\alpha = \beta d\beta;$$

whence

$$x\beta = -y\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From equations (1) and (3), which are of the first degree with respect to α and β, we find

$$\alpha = \frac{x(x^2+y^2)}{2(x^2-y^2)}, \qquad \beta = -\frac{y(x^2+y^2)}{2(x^2-y^2)};$$

and substituting in equation (2), we have

$$(x^2+y^2)^3 = 4a^2(x^2-y^2)^2$$

for the equation of the envelope, which is therefore a lemniscate (see Art. 304).

The Evolute Regarded as an Envelope.

364. We have seen (Art. 332) that the evolute of a curve is tangent to all the normals. Hence, if the equation of the normal to a given curve is expressed in terms of a single parameter, the evolute may be found as an envelope.

For example, the coordinates of a point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

when expressed in terms of an auxiliary variable, are

$$x = a \sec \psi, \qquad y = b \tan \psi;$$

hence the equation of the normal at this point is

$$y - b \tan \psi = -\frac{a}{b} \sin \psi (x - a \sec \psi),$$

or

$$by + ax \sin \psi = (a^2 + b^2) \tan \psi, \quad . \quad . \quad . \quad (1)$$

in which ψ serves as an arbitrary parameter for the system of normals. Taking the derivative with respect to ψ,

$$ax \cos \psi = (a^2 + b^2) \sec^2 \psi,$$

or

$$x = \frac{a^2 + b^2}{a} \sec^3 \psi. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting this value of x in equation (1), we find

$$y = -\frac{a^2 + b^2}{b} \tan^3 \psi. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating ψ between equations (2) and (3), we have for the envelope

$$(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}. \quad . \quad . \quad . \quad . \quad (4)$$

Envelopes of Straight-line Systems.

365. It should be noticed that, whenever the system is one of straight lines, each of the equations $f(x, y, \alpha) = 0$ and $f'(x, y, \alpha) = 0$ is of the first degree with respect to x and y; and therefore it will always be possible to express x and y in terms of α as a third variable, although the final elimination giving the rectangular equation may not be practicable.

For example, let us find the envelope of the reflected rays when parallel rays of light fall on a semicircular mirror. Let AP, Fig. 70, a ray parallel to the axis of x, meet the circle at P, where the angle POR at the centre is α. Then, because the angles of incidence and reflection APO and OPR are equal, the inclination of PR to the axis of x is 2α. It follows that the intercept $OR = \frac{1}{2}a \sec \alpha$. Hence the equation of the reflected ray may be written

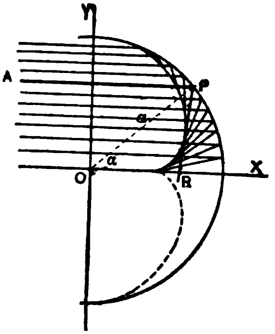

FIG. 70.

$$y \cot 2\alpha = x - \tfrac{1}{2}a \sec \alpha. \quad . \quad . \quad . \quad . \quad (1)$$

The derivative with respect to α is

$$\frac{2y}{\sin^2 2\alpha} = \frac{a \sin \alpha}{2 \cos^2 \alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence

$$y = a \sin^3 \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and substituting in equation (1),

$$x = \tfrac{1}{2}a \cos \alpha (3 - 2 \cos^2 \alpha). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Comparing these equations with Art. 282, we see that the envelope is the two-cusped epicycloid in which the radius of the fixed circle is $\frac{1}{2}a$ and that of the rolling circle is $\frac{1}{4}a$.

Envelopes of this character are called *caustics*, and a cusp like that at $(\frac{1}{2}a, 0)$, the limiting position of R in Fig. 70, is called the *focus* (burning point) because a large number of reflected rays pass very nearly through it.

366. Another example of the envelope of a system of straight lines is the curve of which a given curve is the pedal.

In Fig. 66, p. 330, if the curve BPC is given, the locus

DRE of *R*, the foot of the perpendicular upon the tangent from a fixed point *O*, is called *the pedal* of *BPC*. Conversely, when *DRE* is given, *BPC* is called its *negative pedal* with respect to the pole *O*. Thus the negative pedal is *the envelope of the perpendicular to the radius vector OR at its extremity*.

Now using r and θ to denote the polar coordinates of the point *R* on the given curve, the equation of the perpendicular will be

$$x \cos \theta + y \sin \theta = r, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which we now regard θ as the arbitrary parameter, while r is a known function of θ. Taking the derivative, we have

$$- x \sin \theta + y \cos \theta = \frac{dr}{d\theta}, \quad . \quad . \quad . \quad . \quad (2)$$

and eliminating y and x successively, we obtain

$$\left.\begin{aligned} x &= r \cos \theta - \frac{dr}{d\theta} \sin \theta, \\ y &= r \sin \theta + \frac{dr}{d\theta} \cos \theta, \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the rectangular coordinates of the negative pedal.

367. For example, let it be required *to determine the negative pedal of the strophoid, the node being the pedal origin.*

The polar equation of the strophoid referred to its node, found by transforming equation (1), Art. 257 (and reversing the direction of the initial line), is

$$r = \frac{a \cos 2\theta}{\cos \theta} = a (\cos \theta - \sin \theta \tan \theta);$$

whence

$$\frac{dr}{d\theta} = - a \sin \theta(2 + \sec^2\theta).$$

Substituting in equations (3) of the preceding article, and reducing, we have

$$x = a \sec^2\theta \qquad \text{and} \qquad y = -2a \tan\theta;$$

whence, eliminating θ,

$$y^2 = 4a(x - a).$$

Hence the negative pedal is a parabola whose vertex is situated at the point $(a, 0)$, the vertex of the strophoid.

Examples XXIX.

1. Find the envelope of the system of parabolas represented by the equation

$$y^2 = \frac{\alpha^3}{c}(x - \alpha),$$

in which α is an arbitrary parameter and c a fixed constant.

$$y^2 = \frac{4}{27c}x^3.$$

2. Find the envelope of the circles described on the double ordinates of an ellipse as diameters. $\dfrac{x^2}{a^2 + b^2} + \dfrac{y^2}{b^2} = 1.$

3. Find the envelope of the ellipses, the product of whose semi-axes is equal to the constant c^2.

The conjugate hyperbolas, $2xy = \pm c^2$.

4. Find the envelope of a perpendicular to the normal to the parabola, $y^2 = 4ax$, drawn through the intersection of the normal with the axis. $y^2 = 4a(2a - x)$.

5. Show that the hyperbolas given by the equation

$$\frac{\alpha}{x} + \frac{\beta}{y} = 1, \qquad \text{when} \qquad \alpha + \beta = c,$$

form a pencil, and therefore do not admit of an envelope.

6. A circle moves with its centre on a parabola whose equation is $y^2 = 4ax$, and passes through the vertex of the parabola; find the envelope. The cissoid, $y^2(x + 2a) + x^3 = 0$.

7. A straight line cuts the coordinate axes in such a manner that the product of the intercepts is constant and equal to c^2; find the envelope. $xy = \frac{1}{4}c^2$.

8. A perpendicular to the tangent to a parabola is drawn at the point where the tangent cuts the fixed line $x = c$; find the equation of the envelope of this perpendicular.

The parabola, $y^2 = -4c(x - a - c)$.

9. Circles are described upon the double ordinates of the parabola $y^2 = 4ax$; determine what portion of them fail to touch the envelope of the system.

10. Find the curve to which $y = mx + \frac{a}{m}$ is tangent.

$y^2 = 4ax$.

11. The centre of an hyperbola passing through the origin, and having asymptotes parallel to the axes, moves upon the circle $x^2 + y^2 = a^2$; find the envelope. $x^2y^2 = a^2(x^2 + y^2)$.

12. Find the envelope of the parabolas

$$\frac{x^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} + \frac{y^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} = 1$$

(which touch the coordinate axes at the distances α and β from the origin), when $\alpha\beta = c^2$. $16xy = c^2$.

13. The intercepts α and β of a straight line on any two coordinate axes are connected by the linear relation

$$n\alpha + \beta = c;$$

find the envelope.

The parabola, $(y - nx)^2 - 2ncx - 2cy + c^2 = 0$.

14. Find the envelope of the system of curves,

$$\frac{x^n}{\alpha^n} + \frac{y^n}{\beta^n} = 1,$$

when

$$\alpha^m + \beta^m = c^m.$$

$$x^{\frac{mn}{m+n}} + y^{\frac{mn}{m+n}} = c^{\frac{mn}{m+n}}.$$

15. From a point in the ellipse perpendiculars are drawn to the axes; find the envelope of the line joining the feet of these perpendiculars.

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

16. A straight line of fixed length a moves with its extremities on the two rectangular coordinate axes; determine the envelope.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

17. Let a perpendicular to the line AB in example 16 be drawn through the extremity which slides in the axis of x; find the envelope, and prove that it is one of the involutes of the astroid whose semi-axis is $2a$.

$$\left.\begin{aligned} x &= a \sin^2\alpha \cos\alpha + a \cos\alpha, \\ y &= a \sin\alpha \cos^2\alpha. \end{aligned}\right\}$$

18. Find the envelope of the circles whose centres are on the fixed circle $x^2 + y^2 = a^2$ and which touch the axis of x.

The two-cusped epicycloid $4(x^2 + y^2 - a^2)^3 = 27a^4y^2$.

19. Find the equation of the evolute of the parabola $y^2 = 4ax$, using the equation of the normal in terms of its direction-ratio, viz.,

$$y = mx - 2am - am^3.$$

$$27ay^2 = 4(x - 2a)^3.$$

20. Find the equation of the evolute of the cycloid, by means of the equation of the normal in terms of ψ.

The equation of the normal is

$$x + y\frac{\sin\psi}{1 - \cos\psi} - a\psi = 0.$$

The equations of the evolute are

$$x = a(\psi + \sin\psi) \quad \text{and} \quad y = -a(1 - \cos\psi).$$

Compare Art. 339.

21. From any point C on the circumference of a circle whose radius is a an ordinate to the fixed diameter AB is drawn, and through the foot of the ordinate a perpendicular to the chord AC is drawn; find the equations of the envelope of this perpendicular, and trace the curve.

Denoting by θ the angle BAC, and taking the origin at A, we derive

$$\left.\begin{aligned} x &= 2a\cos^2\theta(3-2\cos^2\theta), \\ y &= -4a\sin\theta\cos^3\theta. \end{aligned}\right\}$$

The curve is symmetrical to the axis of x. $\theta = 0$ gives the point B, $\theta = 30°$ gives a cusp, and $\theta = 90°$ a cusp at the origin.

22. In the figure of Ex. 21, the foot of the ordinate is joined to the middle point of AC; find the envelope, and show that it is the same as the result of Ex. 21, with A and B interchanged.

$$\left.\begin{aligned} x &= 2a\cos^2\theta\cos 2\theta, \\ y &= 2a\sin^2\theta\sin 2\theta. \end{aligned}\right\}$$

The curve is the three-cusped hypocycloid.

23. Determine the negative pedal (see Art. 366) of the parabola $y^2 = 4ax$ with respect to its vertex.

The semicubical parabola $(x-4a)^3 = 27ay^2$.

24. Determine the negative pedal of the cissoid

$$r = 2a\frac{\sin^2\theta}{\cos\theta}.$$

$$y^2 = -8ax.$$

25. Prove that the negative pedal of the spiral of Archimedes is the involute of the circle.

26. Show that the negative pedal of the curve $r = b\sin m\theta$ is determined by

$$\left.\begin{aligned} x &= b\sin m\theta\cos\theta - mb\cos m\theta\sin\theta, \\ y &= b\sin m\theta\sin\theta + mb\cos m\theta\cos\theta, \end{aligned}\right\}$$

being a hypocycloid or an epicycloid, according as m is greater or less than unity.

27. Find the caustic of the circle when the incident rays proceed from a point on the circumference.

The cardioid $\left.\begin{aligned} x &= \tfrac{3}{4}a\cos 2\theta - \tfrac{1}{4}a\cos 4\theta, \\ y &= \tfrac{3}{4}a\sin 2\theta - \tfrac{1}{4}a\sin 4\theta. \end{aligned}\right\}$

28. Derive the polar equation of the negative pedal of the curve

$$r^m = a^m \cos m\theta.$$

The rectangular equations are

$$x = a(\cos m\theta)^{\frac{1-m}{m}} \cos (1 - m)\theta, y = a(\cos m\theta)^{\frac{1-m}{m}} \sin (1 - m)\theta.$$

Whence, denoting the polar coordinates of the pedal by r' and θ',

$$\tan \theta' = \frac{y}{x} = \tan (1 - m)\theta, \qquad \text{and} \qquad r' = a(\cos m\theta)^{\frac{1-m}{m}}.$$

Therefore, eliminating θ,

$$r'^{\frac{m}{1-m}} = a^{\frac{m}{1-m}} \cos \frac{m\theta'}{1 - m}.$$

CHAPTER VIII.

FUNCTIONS OF TWO OR MORE VARIABLES.

XXX.

Partial Differentials.

368. When u denotes a function of the single variable x, the differentials which measure at any instant the rates of variation of x and u are connected by the equation

$$du = f'(x)dx,$$

where $f'(x)$ is the derivative of u with respect to x. Accordingly, when $\frac{du}{dx}$ is used as the symbol of this derivative, du in the numerator is the measure of the rate of u due to the variation of x. Now if u is also a function of y, the du in the symbol $\frac{du}{dy}$ for the derivative with respect to y is a new quantity, having a like relation to the variation of y. These two quantities, which we may distinguish at present by the symbols d_xu and d_yu, are called *partial differentials* of u, while du, which measures the actual rate of u when both x and y vary, is called *the total differential* of u.

369. We have seen in Art. 86 that, as a consequence of the rules of differentiation, the total differential of any function expressible in terms of the elementary functions is the sum of the partial differentials; or, in the present notation,

$$du = d_x u + d_y u.$$

We shall now show that this is true for all continuous functions, as a consequence of the relation between differentials and finite differences.

Let

$$u = f(x, y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$u' = f(x + \Delta x, y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$u'' = f(x + \Delta x, y + \Delta y); \quad . \quad . \quad . \quad . \quad (3)$$

then if $\Delta_x u$ denotes the increment due to increasing x by Δx, and a like meaning is given to $\Delta_y u$, while Δu indicates the increment due to an increase in each variable, we have

$$\Delta_x u = u' - u, \qquad \Delta_y u' = u'' - u, \qquad \Delta u = u'' - u,$$

whence

$$\Delta u = \Delta_x u + \Delta_y u'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Dividing each member by Δt, the increment of time corresponding to Δx and Δy, we have

$$\frac{\Delta u}{\Delta t} = \frac{\Delta_x u}{\Delta t} + \frac{\Delta_y u'}{\Delta t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now, by Art. 24, each of these ratios has for its limit the corresponding rate; in other words, it differs from the rate by a quantity e which vanishes with Δt. Thus we may write

$$\frac{du}{dt} + e = \frac{d_x u}{dt} + e' + \frac{d_y u'}{dt} + e'', \quad . \quad . \quad . \quad (6)$$

in which e, e' and e'' vanish with Δt. But, when $\Delta t = 0$, $\Delta x = 0$ and, by equation (2), $u' = u$; hence, when $\Delta t = 0$, equation (6) becomes

$$\frac{du}{dt} = \frac{d^xu}{dt} + \frac{d_yu}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Therefore

$$du = d_xu + d_yu. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This principle is readily extended, by the same method, to functions of several independent variables.

Partial Derivatives.

370. The theorem of partial differentials is sometimes written

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which the coefficients of dx and dy (which are the derivatives of u with respect to x and to y) are called *partial derivatives*. Their numerators, though identical in form, are, as shown in Art. 368, really equal to d_xu and d_yu respectively. The absence of distinctive marks will however produce no confusion, if it is understood that the fractional form $\frac{du}{dx}$ will be used only to express the derivative of u with respect to x, no matter how many independent variables may be under consideration.

371. If x and y are made functions of any other variable t, u becomes a function of t, and equation (1) gives

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which the derivative of u with respect to t is expressed in terms of the partial derivatives due to its expression as a function of the two independent variables* x and y.

Geometrical Representation of Partial Derivatives.

372. Let the independent variables x and y be rectangular coordinates of a point R in the horizontal plane, and let u be the vertical third coordinate of a point P in space; then

$$u = f(x, y) \qquad \dots\dots \qquad (1)$$

is the equation of a surface. If a plane parallel to the plane of xz be passed through PR, y will have a constant value in this plane; hence when y is regarded as a constant, equation (1) will become the equation of the intersection of this plane with the surface. It follows that if ψ_1 is the inclination to the plane of xy of the tangent at P to this curve, tan ψ_1 will represent the derivative of u with respect to x. Denoting, in like manner, by ψ_2 the inclination of a line tangent to the section of the surface made by a plane parallel to that of yz, we have

$$\tan \psi_1 = \frac{du}{dx}, \qquad \tan \psi_2 = \frac{du}{dy}.$$

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* If x is identical with the new variable t (which is as much as to suppose that y is given as an explicit function of x), the equation becomes

$$\left(\frac{du}{dx}\right) = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx},$$

in which a special mark must be used to distinguish *the total derivative* of u with respect to x (after y is eliminated) from the partial derivative in which y is regarded as independent. This distinction is sometimes made by adopting a different form of the letter d in the partial derivatives; thus

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

373. At all ordinary points of the surface, the plane which passes through these two tangent lines is a tangent plane to the surface. Now, when R moves in any manner due to the variation of x and y, P moves in some curve drawn in the surface, and the tangent to its path lies in this tangent plane. It follows, as in Art. 37 (see Fig. 7, p. 27), that the increment of u, *as measured up to this tangent plane*, represents the hypothetical increment which is denoted by du in accordance with Art. 23.

If ds is the differential of the actual motion of R in the horizontal plane, the total differential du corresponds to the motion ds, and the partial differentials d_xu and d_yu correspond to the resolved parts of this motion in the directions of the axes. The u-coordinates of the tangent plane, corresponding to the four corners of the rectangle whose sides are dx and dy, are u, $u + d_xu$, $u + d_yu$, and $u + du$. These form the edges of a truncated prism, and it is readily proved geometrically that the sums of opposite edges are equal. Hence $du = d_xu + d_yu$.

374. The derivative $\dfrac{du}{ds} = \tan\psi$, where ψ is the inclination to the horizontal plane of the tangent to the actual path of P. It is called a "total derivative" only with reference to the partial derivative theorem (equation (2), Art. 371), which gives its value in terms of derivatives with respect to x and y, namely,

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} + \frac{du}{dy}\frac{dy}{ds}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If ϕ denotes, as usual, the inclination of ds in the plane of xy to the axis of x, equation (1) becomes

$$\frac{du}{ds} = \frac{du}{dx}\cos\phi + \frac{du}{dy}\sin\phi.$$

This is, in effect, a relation between the trigonometric tangents of the inclinations to the horizontal plane of the three tangent lines in the surface; that is to say of the angles we have denoted ψ, ψ_1 and ψ_2.*

Higher Derivatives.

375. As in Art. 97, we may regard $\dfrac{d}{dx}$ and $\dfrac{d}{dy}$ as symbols of the operation of taking the derivative, which may be detached from the operand u. Single letters may also be used as in Art. 101, a distinctive mark being employed, in this case, to indicate the independent variable. Thus we may put $D = \dfrac{d}{dx}$ and $D' = \dfrac{d}{dy}$.

The partial derivative of u with respect to either variable will in general be a function of both variables, and will therefore admit of a derivative with respect to either variable. These derivatives are called the derivatives of u of the second order. Thus we have the four derivatives

$$\frac{d}{dx}\frac{du}{dx}, \qquad \frac{d}{dy}\frac{du}{dx}, \qquad \frac{d}{dx}\frac{du}{dy}, \qquad \frac{d}{dy}\frac{du}{dy}.$$

Using the abbreviated notation, introduced in Art. 97 in the case of a single independent variable, these are usually written

$$\frac{d^2u}{dx^2}, \qquad \frac{d^2u}{dy\,dx}, \qquad \frac{d^2u}{dx\,dy}, \qquad \frac{d^2u}{dy^2},$$

in which no separate meaning has been given to the numerators, and the order of the operations is that of the factors in the denominator.

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* This relation may be verified by spherical trigonometry.

Again, they may be written in the symbolic form

$$D^2u, \qquad D'Du, \qquad DD'u, \qquad D'^2u.$$

376. If we use Δ to denote the operation of taking the finite or actual difference between two values of a function, consequent upon an increment of one variable, we may use $\frac{\Delta}{\Delta x}$ and $\frac{\Delta}{\Delta y}$ for the operation of taking such a difference and then dividing by the increment upon which it depends. Then, corresponding to every derivative of the second or a higher order, we have a corresponding combination of these symbols. We shall now show that each of the higher derivatives is the limit of the value of the corresponding Δ-symbol when Δx and Δy simultaneously vanish.

377. For this purpose, let us resume the consideration of the equation of Art. 40,

$$\frac{\Delta u}{\Delta x} = \frac{du}{dx} + e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in connection with which this relation is expressed, for the first derivative, by the statement that e vanishes with Δx. Now this quantity e is a function not only of Δx, but of the independent variables x and y. Hence it admits of differentiation with respect to each of these variables. But, since e always assumes the value zero when $\Delta x = 0$, it is, for this value of Δx, *independent of x and y*. Therefore the derivatives $\frac{de}{dx}$ and $\frac{de}{dy}$ vanish with Δx.

378. Equation (1) shows that, in order to apply the symbol $\frac{\Delta}{\Delta x}$ to a function of x, we must change Δ to d and then add a quantity which vanishes with Δx. Thus, applying the operation to equation (1) itself, we have

$$\begin{aligned}\frac{\Delta}{\Delta x}\frac{\Delta u}{\Delta x} &= \frac{\Delta}{\Delta x}\left(\frac{du}{dx}+e\right)\\ &= \frac{d}{dx}\left(\frac{du}{dx}+e\right)+e'\\ &= \frac{d^2u}{dx^2}+\frac{de}{dx}+e',\end{aligned}$$

in which both $\frac{de}{dx}$ and e' vanish with Δx; so that we may write

$$\frac{\Delta^2 u}{\Delta x^2} = \frac{d^2u}{dx^2}+e. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, applying the operation $\frac{\Delta}{\Delta y}$ to equation (1), we find

$$\begin{aligned}\frac{\Delta}{\Delta y}\frac{\Delta u}{\Delta x} &= \frac{d}{dy}\left(\frac{du}{dx}+e\right)+e''\\ &= \frac{d^2u}{dy\,dx}+\frac{de}{dy}+e'',\end{aligned}$$

in which $\frac{de}{dy}$ vanishes with Δx and e'' vanishes with Δy; so that we may write

$$\frac{\Delta^2 u}{\Delta y\,\Delta x} = \frac{d^2u}{dy\,dx}+e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where e vanishes when both Δx and Δy vanish.

In like manner, it may be shown that the Δ-symbols of higher orders, whether there are two or more independent variables concerned, have for their limits the corresponding higher derivatives.

Commutative Character of Differentiation.

379. We have hitherto attached no meaning to the separate terms of the symbols in fractional form employed in the preceding articles. But with respect to the Δ-symbols this is readily done. Thus, in equation (2), the first member is the ratio to Δx of $\Delta\frac{\Delta u}{\Delta x}$. This last denotes the increment of a fraction whose denominator is constant, which is obviously the same as the result of dividing the increment of the numerator by the denominator; that is,

$$\Delta\frac{\Delta u}{\Delta x} = \frac{\Delta . \Delta u}{\Delta x}.$$

Thus $\Delta^2 u$ in equation (2) is simply the abbreviated symbol for the increment of Δu due to the increment Δx in the independent variable x.

380. With respect to the symbol $\Delta^2 u$ in equations (2) and (3), a distinction must be made between increments due to Δx and to Δy respectively. Using suffixes for this purpose, the $\Delta^2 u$ in equation (3) is an abbreviated form of $\Delta_y(\Delta_x u)$. Now, putting

$$u = f(x, y),$$

we have, as in Art. 369,

$$\Delta_x u = f(x + \Delta x, y) - f(x, y);$$

if in this equation we replace y by $y + \Delta y$, we obtain a new value of $\Delta_x u$, and, denoting this value by $\Delta'_x u$, we have

$$\Delta'_x u = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y).$$

Since this change in the value of $\Delta_x u$ results from the increment received by y, the increment received by $\Delta_x u$ will be $\Delta_y(\Delta_x u)$; hence

$$\Delta_y(\Delta_x u) = \Delta'_x u - \Delta_x u,$$

or

$$\Delta_y(\Delta_x u) = f(x+\Delta x,\ y+\Delta y) - f(x,\ y+\Delta y) - f(x+\Delta x,\ y) + f(x,y).$$

381. The value of $\Delta_x(\Delta_y u)$, obtained in a precisely similar manner, is identical with that just given; hence

$$\Delta_y(\Delta_x u) = \Delta_x(\Delta_y u). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It follows that the values of the fractions

$$\frac{\Delta_y\,\Delta_x u}{\Delta y\,\Delta x} \quad \text{and} \quad \frac{\Delta_x\,\Delta_y u}{\Delta x\,\Delta y}$$

are the same; whence we infer that the derivatives which are their limiting values when Δx and Δy vanish are also equal. Thus

$$\frac{d^2u}{dy\,dx} = \frac{d^2u}{dx\,dy}.$$

The theorem expressed by this equation is readily verified in any particular case. Thus, given

$$u = y^x,$$

we have

$$\frac{du}{dx} = y^x \log y, \qquad \frac{du}{dy} = xy^{x-1};$$

taking the derivative of the first with respect to y, and that of the second with respect to x, we obtain the same expression, namely,

$$y^{x-1}(x \log y + 1).$$

382. The result arrived at above may be expressed thus: *the operations of taking the derivative with respect to two independent variables are commutative;* that is, they may be interchanged without affecting the result obtained.

This theorem may be extended to derivatives higher than the second, and also to functions of more than two independent variables. For it has been proved that we may, without affecting the result, interchange any two consecutive differentiations, and it is obvious that, by successive interchanges of consecutive differentiations, we can alter the order of differentiation in any manner desired. Hence all differentiations with respect to independent variables are commutative.

Accordingly, the result of differentiating m times with respect to x, n times with respect to y, and p times with respect to z, may, without regard to the order of differentiation, be expressed by the symbol

$$\frac{d^{m+n+p}u}{dx^m dy^n dz^p}.$$

Commutative and Distributive Operations.

383. When m is constant we have

$$d(mu) = mdu;$$

hence when u is a function of x

$$\frac{d}{dx}mu = m\frac{d}{dx}u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation indicates that the operation of multiplying by a constant and the operation of taking the derivative with reference to x are commutative. It follows that the factors of the symbolic product $y\frac{d}{dx}$, when y is a variable independent of x, are commutative; for, in performing the operation $\frac{d}{dx}$, y is regarded as a constant.

On the other hand, $x\frac{d}{dx}$ is not commutative; for

$$\frac{d}{dx}xu = x\frac{du}{dx} + u,$$

while

$$x\frac{d}{dx}u = x\frac{du}{dx}.$$

384. A repeated application of the same symbol, whether simple or compound, is indicated by affixing an index to the symbol; thus,

$$\left(y\frac{d}{dx}\right)^2 = y\frac{d}{dx}\cdot y\frac{d}{dx};$$

and, since the operations indicated by the symbol are commutative, we have

$$\left(y\frac{d}{dx}\right)^2 = y^2\frac{d^2}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

On the other hand, since the operations indicated by the symbol $x\frac{d}{dx}$ are not commutative, $\left(x\frac{d}{dx}\right)^2$ is *not* equal to $x^2\frac{d^2}{dx^2}$. We have, in fact,

$$\left(x\frac{d}{dx}\right)^2 u = x\frac{du}{dx} + x^2\frac{d^2u}{dx^2}. \quad . \quad . \quad . \quad . \quad (2)$$

385. The result obtained by adding the results of the application of two operative symbols is expressed by prefixing to the operand the sum of the operative symbols. For example, the result written above may be expressed thus:

$$\left(x\frac{d}{dx}\right)^2 = x\frac{d}{dx} + x^2\frac{d^2}{dx^2};$$

for this is a general formula applicable to any function of x as operand.

On the other hand, we may write formulæ involving special forms of the operand, thus

$$\left(x\frac{d}{dx}\right)x^m = mx^m;$$

whence it readily follows that

$$\left(x\frac{d}{dx}\right)^r x^m = m^r x^m.$$

386. An operation which, when applied to a sum, gives the sum of the results of separate application to the parts is said to be *distributive* over a sum. Thus, since

$$\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v,$$

differentiation is a distributive operation. In this formula, the symbol of differentiation is applied to a sum exactly as if it were an ordinary algebraic factor; in like manner, every combination of differential symbols and algebraic multipliers has the same property.

In algebra, an exponent is distributive over the factors of a product, as expressed by the formula $(abc)^m = a^m b^m c^m$. But this is not true of operative symbols *unless the symbolic factors are commutative.* Compare equations (1) and (2), Art. 384.

Symbolic Identities.

387. The formulæ of algebraic expansion are consequences of the commutative and distributive nature of algebraic multiplication; hence it follows that a symbolic product or power may be expanded by these formulæ; provided all the constituent symbols represent commutative operations. Thus

$$\left(\frac{d}{dx} + a\right)\left(\frac{d}{dx} - a\right)u = \frac{d^2u}{dx^2} - a^2u.$$

Again, when the constituent symbols are not commutative, we may obtain symbolic identities by the rules of differentiation. Thus, let u denote a function of θ; then we have, by differentiation,

$$\frac{d}{d\theta}e^{n\theta}u = e^{n\theta}\left(\frac{d}{d\theta} + n\right)u,$$

and multiplying by $e^{-n\theta}$,

$$e^{-n\theta}\frac{d}{d\theta}e^{n\theta}\cdot u = \left(\frac{d}{d\theta} + n\right)u. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Applying now the symbols whose equivalence is expressed in this equation to the equation itself, we have

$$e^{-n\theta}\frac{d}{d\theta}e^{n\theta}\cdot e^{-n\theta}\frac{d}{d\theta}e^{n\theta}\cdot u = e^{-n\theta}\frac{d^2}{d\theta^2}e^{n\theta}\cdot u = \left(\frac{d}{d\theta} + n\right)^2\cdot u;$$

and, by repeating this process, we have in general

$$e^{-n\theta}\frac{d^r}{d\theta^r}e^{n\theta}\cdot u = \left(\frac{d}{d\theta}+n\right)^r\cdot u.^* \quad . \quad . \quad . \quad . \quad (2)$$

Euler's Theorems concerning Homogeneous Functions.

388. A homogeneous algebraic function of the nth degree involving two variables may be put in the form

$$u = x^n f\left(\frac{y}{x}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Any expression which comes under this form, even when f is a transcendental function, and when n has a fractional or negative value, is called a ***homogeneous function.*** Thus

$$u = \frac{\sqrt{x}+\sqrt{y}}{x(x+y)}$$

is a homogeneous function in which $n = -\frac{3}{2}$. Again,

$$u = \log\,[x+\sqrt{(x^2+y^2)}] - \log x$$

is a homogeneous function in which $n = 0$.

389. By differentiating equation (1), we derive

$$\frac{du}{dx} = nx^{n-1}f\left(\frac{y}{x}\right) - x^{n-2}yf'\left(\frac{y}{x}\right),$$

and

$$\frac{du}{dy} = x^{n-1}f'\left(\frac{y}{x}\right);$$

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* This equation is equivalent to the result obtained by means of Leibnitz' theorem in Art. 107.

whence

$$x\frac{du}{dx} + y\frac{du}{dy} = nx^n f\left(\frac{y}{x}\right) = nu,$$

or, symbolically,

$$\left(x\frac{d}{dx} + y\frac{d}{dy}\right)u = nu, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

when u is a homogeneous function of the nth degree.

Again, the derivatives of u are homogeneous functions of the $(n - 1)$th degree; hence, by the theorem expressed in equation (2), we have

$$\left(x\frac{d}{dx} + y\frac{d}{dy}\right)\frac{du}{dx} = (n - 1)\frac{du}{dx}, \text{ and } \left(x\frac{d}{dx} + y\frac{d}{dy}\right)\frac{du}{dy} = (n - 1)\frac{du}{dy};$$

whence, expanding,

$$x\frac{d^2u}{dx^2} + y\frac{d^2u}{dy\,dx} = (n - 1)\frac{du}{dx}, \text{ and } x\frac{d^2u}{dx\,dy} + y\frac{d^2u}{dy^2} = (n - 1)\frac{du}{dy};$$

multiplying by x and y respectively and adding,

$$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = (n - 1)\left[x\frac{du}{dx} + y\frac{du}{dy}\right];$$

hence

$$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = n(n - 1)u. \quad . \quad . \quad (3)$$

The results expressed in equations (2) and (3), and similar results involving higher derivatives, are known as *Euler's Theorems concerning homogeneous functions.*

Examples XXX.

1. Given $u = \log[x + \sqrt{(x^2 + y^2)}]$, prove that

$$\left(x\frac{d}{dx} + y\frac{d}{dy}\right)u = 1.$$

2. Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

$$\frac{du}{dx} + \frac{du}{dy} + \frac{du}{dz} = \frac{3}{x + y + z}.$$

3. Given $u = \sec(y + ax) + \tan(y - ax)$, prove that

$$\frac{d^2u}{dx^2} = a^2\frac{d^2u}{dy^2}.$$

4. Verify the theorem $\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}$ when $u = \sin(xy^3)$.

5. Verify the theorem $\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}$ when $u = \log\tan(ax + y^3)$.

6. Verify the theorem $\frac{d^3u}{dy^2\,dx} = \frac{d^3u}{dx\,dy^2}$ when $u = \tan^{-1}\frac{x}{y}$.

7. Verify the theorem $\frac{d^3u}{dy\,dx^2} = \frac{d^3u}{dx^2\,dy}$ when $u = y\log(1 + xy)$.

8. Given $u = \sin x \cos y$, prove that

$$\frac{d^4u}{dy^2\,dx^2} = \frac{d^4u}{dx^2\,dy^2} = \frac{d^4u}{dx\,dy\,dx\,dy}$$

9. Given $u = \frac{1}{\sqrt{(4ab - c^2)}}$, prove that

$$\frac{d^2}{dc^2}u = \frac{d^2}{da\,db}u.$$

10. Given $u = (x + y)^2$, prove that

$$x\frac{d^2u}{dx^2} + y\frac{a^2u}{dx\,dy} = \frac{du}{dx}.$$

11. Given $u = \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$, prove that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0.$$

12. Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2u}{dx\,dy} + 2\frac{d^2u}{dy\,dz} + 2\frac{d^2u}{dz\,dx} = -\frac{9}{(x+y+z)^2}.$$

Employ the symbol $\left(\frac{d}{dx} + \frac{d}{dy} + \frac{d}{dz}\right)^2 u$, *and see Ex.* 2.

13. Verify Euler's theorems for $u = (x^2+y^2)^{\frac{1}{2}}$, and for $u = \frac{xy}{x+y}$.

14. If the rectangular axes of x and y are turned through the angle ϕ in the case of a surface $z = f(x, y)$, find $\frac{dz}{dx'}$ and $\frac{dz}{dy'}$ in terms of $\frac{dz}{dx}$ and $\frac{dz}{dy}$; and thence show that $\frac{dz}{dx}$ may be regarded as a "total derivative" when $\frac{dz}{dx'}$ and $\frac{dz}{dy'}$ are the partial derivatives.

15. Prove the symbolic equation

$$\frac{d^m}{dx^m} x^{m+p} \frac{d^p}{dx^p} \cdot u = x^p \frac{d^{m+p}}{dx^{m+p}} x^m \cdot u,$$

by showing the identity of the result of each operation upon x^r, and therefore upon any function developable in a series of powers of x.

16. Apply the symbol $a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3}$ to the expression

$$a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2.$$

Result 0.

17. Operate on $a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3$ with the symbol $a_0\dfrac{d}{da_1} + 2a_1\dfrac{d}{da_2} + 3a_2\dfrac{d}{da_3} + 4a_3\dfrac{d}{da_4}$.

Result 0.

18. Determine the value of

$$x\frac{du}{dx} + y\frac{du}{dy}, \qquad \text{when} \qquad u = \tan^{-1}\frac{x^2 - y^2}{ax}.$$

Solution:

Since tan u is a homogeneous function of the first degree, we have, by Euler's theorem,

$$\left(x\frac{du}{dx} + y\frac{du}{dy}\right)\sec^2 u = \tan u;$$

whence

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{ax(x^2 - y^2)}{a^2x^2 + (x^2 - y^2)^2}.$$

19. If $u = \sin v$, v being a homogeneous function of the nth degree, determine the value of $x\dfrac{du}{dx} + y\dfrac{du}{du}$.

$$x\frac{du}{dx} + y\frac{du}{dy} = nv\cos v.$$

XXXI.

Change of the Independent Variable.

390. It is frequently desirable to transform expressions involving derivatives with reference to x into equivalent expressions, in which some variable connected with x by a known relation is the independent variable. This process is called *changing the independent variable.*

Let y denote the function whose derivatives occur in the given expression, and let θ denote the new independent variable. For the first derivative, we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx},$$

in which it is necessary to express $\frac{d\theta}{dx}$ in terms of θ by means of the given relation. For example, suppose this relation to be

$$x = \tan\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

differentiating, $dx = \sec^2\theta\, d\theta$, whence,

$$\frac{d\theta}{dx} = \cos^2\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and we have, in this case,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cos^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

391. To express the second derivative with respect to x in terms of θ, it is necessary to differentiate this expression as a product of variable factors. Thus

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left[\frac{dy}{d\theta}\cos^2\theta\right]\frac{d\theta}{dx}$$

$$= \cos^2\theta\left[\cos^2\theta\frac{d^2y}{d\theta^2} - 2\cos\theta\sin\theta\frac{dy}{d\theta}\right]. \quad . \quad (4)$$

In like manner, expressions for the third and higher derivatives may be obtained.

It will be noticed that, while the result in the case of the first derivative might be obtained by substituting for dx in the denominator its value in terms of θ, the second derivative cannot be found in a similar manner. This is due to the fact that a common meaning cannot be assigned to the symbols 'd^2y' which constitute the numerators of the second derivatives. Compare the expressions of which the second derivatives are the limits, Art. 379.

392. Change of independent variable may be used to simplify differential equations. For example, the differential equation

$$(1 + x^2)^2 \frac{d^2y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} + y = 0$$

is transformed by means of equations (1), (3) and (4) above into the much simpler form

$$\frac{d^2y}{d\theta^2} + y = 0.$$

Transformations Involving Partial Derivatives.

393. Let u denote a function of x and y, and let r and θ be two new independent variables connected with x and y by two given equations, by virtue of which x and y are functions of r and θ. Then u is also a function of r and θ, and it may be required to express the partial derivatives of u with reference to x and y in terms of derivatives with reference to r and θ.

In Art. 371, the two independent variables x and y are supposed to be made functions of a new variable t, and equation (2) shows how to obtain the derivative with respect to this new variable. In finding the required expression for $\frac{du}{dx}$, r and θ are the two independent variables, and x takes the place of t; thus we have

$$\frac{du}{dx}=\frac{du}{dr}\frac{dr}{dx}+\frac{du}{d\theta}\frac{d\theta}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

in which $\frac{dr}{dx}$ and $\frac{d}{dx}$ are to be derived from the given relations between the two pairs of independent variables, just as $\frac{d\theta}{dx}$, in Art. 390, is derived from the single relation there given.

Similarly we have

$$\frac{du}{dy}=\frac{du}{dr}\frac{dr}{dy}+\frac{du}{d\theta}\frac{d\theta}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

in which $\frac{dr}{dy}$ and $\frac{d\theta}{dy}$ are found in like manner.

The four coefficients are, in fact, the derivatives of r and θ when x and y are associated together as a pair of independent variables, but their values are to be expressed in terms of r and θ.

394. For example, let us assume that the given relations are those which connect polar and rectangular coordinates, namely,

$$x=r\cos\theta \qquad \text{and} \qquad y=r\sin\theta. \quad . \quad . \qquad (1)$$

Solving these equations for r and θ, we derive

$$r^2=x^2+y^2 \qquad \text{and} \qquad \theta=\tan^{-1}\frac{y}{x}; \quad . \quad . \qquad (2)$$

whence

$$\left.\begin{aligned}\frac{dr}{dx} &= \frac{x}{\sqrt{(x^2+y^2)}} = \cos\theta, \quad \frac{d\theta}{dx} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r},\\ \frac{dr}{dy} &= \frac{y}{\sqrt{(x^2+y^2)}} = \sin\theta, \quad \frac{d\theta}{dy} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}.\end{aligned}\right\}^{*} \quad . \quad (3)$$

Substituting these values in the general equations of the preceding article, we have

$$\frac{du}{dx} = \frac{du}{dr}\cos\theta - \frac{du}{d\theta}\frac{\sin\theta}{r}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{du}{dy} = \frac{du}{dr}\sin\theta + \frac{du}{d\theta}\frac{\cos\theta}{r}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Partial Derivatives of the Second Order.

395. In finding expressions for the derivatives of the second order, it must be remembered that $\frac{du}{dr}$ and $\frac{du}{d\theta}$, which occur in the second members of equations (1) and (2), Art. 393, are themselves functions of r and θ. Thus, by equation (2), Art. 371,

$$\frac{d}{dx}\left(\frac{du}{dr}\right)^{\dagger} = \frac{d^2u}{dr^2}\frac{dr}{dx} + \frac{d^2u}{dr\,d\theta}\frac{d\theta}{dx}.$$

---

[*] It should be noticed that these values cannot be found by differentiating equations (1) with respect to r and θ; but $\frac{dr}{dx}$ and $\frac{d\theta}{dx}$ could be found by elimination from the partial derivatives of these equations with respect to x, which are

$$1 = \cos\theta\frac{dr}{dx} - r\sin\theta\frac{d\theta}{dx}$$

and

$$0 = \sin\theta\frac{dr}{dx} + r\cos\theta\frac{d\theta}{dx}.$$

[†] This expression should not be written $\frac{d^2u}{dx\,dr}$, because that would indicate a

In like manner, we have

$$\frac{d}{dx}\left(\frac{du}{d\theta}\right) = \frac{d^2u}{dr\,d\theta}\frac{dr}{dx} + \frac{d^2u}{d\theta^2}\frac{d\theta}{dx}.$$

Now, making use of these results, we derive from equation (1), Art. 393,

$$\frac{d^2u}{dx^2} = \frac{du}{dr}\frac{d^2r}{dx^2} + \frac{dr}{dx}\left[\frac{d^2u}{dr^2}\frac{dr}{dx} + \frac{d^2u}{dr\,d\theta}\frac{d\theta}{dx}\right]$$
$$+ \frac{du}{d\theta}\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx}\left[\frac{d^2u}{dr\,d\theta}\frac{dr}{dx} + \frac{d^2u}{d\theta^2}\frac{d\theta}{dx}\right]$$

or

$$\frac{d^2u}{dx^2} = \frac{du}{dr}\frac{d^2r}{dx^2} + \frac{du}{d\theta}\frac{d^2\theta}{dx^2}$$
$$+ \frac{d^2u}{dr^2}\left(\frac{dr}{dx}\right)^2 + 2\frac{d^2u}{dr\,d\theta}\frac{dr}{dx}\frac{d\theta}{dx} + \frac{d^2u}{d\theta^2}\left(\frac{d\theta}{dx}\right)^2.$$

In like manner, expressions for the derivatives $\frac{d^2u}{dx\,dy}$ and $\frac{d^2u}{dy^2}$ can be found.

396. In applying this general expression to a special case, the additional coefficients $\frac{d^2r}{dx^2}$ and $\frac{d^2\theta}{dx^2}$ are, like $\frac{dr}{dx}$ and $\frac{d\theta}{dx}$, to be derived from the given relations between the pairs of independent variables. For example, when the relations are those given in Art. 394, we derive, from the first two of the expressions included in equations (3),

$$\frac{d^2r}{dx^2} = \frac{d}{dx}\frac{x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{\sin^2\theta}{r}$$

---

partial derivative of *u* when expressed as a function of *r* and *x* regarded as two independent variables.

and

$$\frac{d^2\theta}{dx^2} = \frac{d}{dx}\frac{-y}{x^2+y^2} = \frac{2xy}{(x^2+y^2)^2} = \frac{2\sin\theta\cos\theta}{r^2}.$$

Substituting these values and those before found in the general expression above, we obtain

$$\frac{d^2u}{dx^2} = \frac{\sin^2\theta}{r}\frac{du}{dr} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{du}{d\theta}$$

$$+ \cos^2\theta\frac{d^2u}{dr^2} - 2\frac{\sin\theta\cos\theta}{r}\frac{d^2u}{dr\,d\theta} + \frac{\sin^2\theta}{r^2}\frac{d^2u}{d\theta^2}. \quad (1)$$

397. The transformation of the expression

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}$$

from rectangular to polar coordinates is of importance in mathematical physics.

Since the effect of putting $\frac{1}{2}\pi - \theta$ in place of θ in equations (1), Art. 394, is to interchange x and y, the expression for $\frac{d^2u}{dy^2}$ may, in this case, be derived from equation (1), Art. 396, by interchanging $\sin\theta$ and $\cos\theta$ and reversing the sign of $d\theta$. Hence

$$\frac{d^2u}{dy^2} = \frac{\cos^2\theta}{r}\frac{du}{dr} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{du}{d\theta} +$$

$$\sin^2\theta\frac{d^2u}{dr^2} + 2\frac{\sin\theta\cos\theta}{r}\frac{d^2u}{dr\,d\theta} + \frac{\cos^2\theta}{r^2}\frac{d^2u}{d\theta^2}. \quad (2)$$

Adding equations (1) and (2),

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}. \quad . \quad . \quad . \quad (3)$$

398. The corresponding symbol in three dimensions, when u is also a function of the third rectangular coordinate z, has been denoted by ∇^2; thus,

$$\nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}.$$

Then, by equation (3),

$$\nabla^2u=\frac{d^2u}{dz^2}+\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}. \quad . \quad . \quad . \quad (4)$$

Now z and r constitute rectangular coordinates of P in the plane POZ, where OZ is the axis of z. Hence, denoting OP by ρ and POZ by ϕ, ρ and ϕ are polar coordinates of P in this plane, and

$$z=\rho\cos\phi, \qquad r=\rho\sin\phi. \quad . \quad . \quad . \quad . \quad (5)$$

Therefore equation (3) gives

$$\frac{d^2u}{dz^2}+\frac{d^2u}{dr^2}=\frac{d^2u}{d\rho^2}+\frac{1}{\rho}\frac{du}{d\rho}+\frac{1}{\rho^2}\frac{d^2u}{d\phi^2}. \quad . \quad . \quad . \quad (6)$$

Also by equation (5), Art. 394 (since r in equations (5) takes the place of y),

$$\frac{du}{dr}=\frac{du}{d\rho}\sin\phi+\frac{du}{d\phi}\frac{\cos\phi}{\rho}. \quad . \quad . \quad . \quad . \quad (7)$$

Substituting from equations (6) and (7) and eliminating r, equation (4) becomes

$$\nabla^2 u = \frac{d^2u}{d\rho^2} + \frac{2}{\rho}\frac{du}{d\rho} + \frac{1}{\rho^2}\frac{d^2u}{d\phi^2} + \frac{\cot\phi}{\rho^2}\frac{du}{d\phi} + \frac{1}{\rho^2\sin^2\phi}\frac{d^2u}{d\theta^2}. \quad (8)$$

Equation (4) gives $\nabla^2 u$ in the "cylindrical" coordinates r, θ, z; and equation (8), in the spherical coordinates ρ, θ, ϕ, where

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

Symbolic and Extended Forms of Taylor's Theorem.

399. A polynomial symbol involving D, where $D = \frac{d}{dx}$, may be written in the form $F(D)$ where F is a given rational integral function. If $F(D)$ is a function which can be developed in powers of D, it is regarded as equivalent to this series of symbols. Thus, by the exponential series,

$$e^{hD}f(x) = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right]f(x)$$

$$= f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \cdots,$$

which is by Taylor's theorem the expansion of $f(x+h)$ in powers of h. Hence Taylor's theorem can be expressed in the symbolic form

$$f(x+h) = e^{hD}f(x). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

400. When the operand is a function of y also, this gives

$$f(x+h, y) = e^{hD}f(x, y);$$

again, putting $D'=\dfrac{d}{dy}$, we have in like manner

$$f(x+h,\ y+k)=e^{kD'}e^{hD}f(x,\ y). \quad . \quad . \quad . \quad (2)$$

Now the equation

$$e^{kD'}e^{hD}=e^{hD+kD'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is an algebraic identity for D and D', that is, it shows that the product of the two series represented by the first member is equivalent to the single series represented by the second member. Therefore, because D and D' are commutative symbols, equation (3) is, by Art. 387, a symbolic identity. Thus equation (2) becomes

$$f(x+h,\ y+k)=e^{h\frac{d}{dx}+k\frac{d}{dy}}f(x,\ y),$$

which, when written out in full, takes the form

$$f(x+h,\ y+k)=f(x,\ y)+\frac{df}{dx}h+\frac{df}{dy}k$$
$$+\frac{1}{2!}\left[\frac{d^2f}{dx^2}h^2+2\,\frac{d^2f}{dx\,dy}hk+\frac{d^2f}{dy^2}k^2\right]+\cdots$$

The result is readily extended to any number of independent variables.

Lagrange's Theorem.

401. Let y be an implicit function of the independent variables x and z, satisfying the relation

$$y=z+x\phi(y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which ϕ denotes any function; then, if we have

$$u=f(y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

u will also be a function of x and z.

If now it be required to develop u in a series involving powers of x, we obtain by the application of Maclaurin's theorem

$$u=u_0+\frac{du}{dx}\Big]_0 x+\frac{d^2u}{dx^2}\Big]_0 \frac{x^2}{2!}+\cdots, \quad . \quad . \quad . \quad (3)$$

in which the coefficients are functions of z but not of x. We proceed to transform the derivatives $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, etc. into expressions in which z is the independent variable, before determining their values when $x=0$.

Differentiating equation (1), we obtain the partial derivatives

$$\frac{dy}{dx}=\frac{\phi(x)}{1-x\phi'(y)}, \qquad \frac{dy}{dz}=\frac{1}{1-x\phi'(y)};$$

hence, by equation (2),

$$\frac{du}{dx}=\frac{f'(y)\phi(y)}{1-x\phi'(y)} \quad \text{and} \quad \frac{du}{dz}=\frac{f'(y)}{1-x\phi'(y)}; \quad . \quad . \quad (4)$$

whence

$$\frac{du}{dx}=\phi(y)\frac{du}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

402. In order to deduce the required expressions for the higher derivatives, we first establish the following general theorem:

When y is a function of x and z, and u and $\psi(y)$ are any functions of y, we have

$$\frac{d}{dx}\psi(y)\frac{du}{dz}=\frac{d}{dz}\psi(y)\frac{du}{dx}. \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

To prove this theorem, we have only to perform the differentiations; thus, putting $f(y)$ for u, each member of (6) reduces to

$$\psi'(y)f'(y)\frac{dy}{dx}\frac{dy}{dz}+\psi(y)\frac{d^2u}{dx\,dz}.$$

Substituting, in the general theorem (6), the value of $\frac{du}{dx}$ given in equation (5), we have

$$\frac{d}{dx}\psi(y)\frac{du}{dz}=\frac{d}{dz}\phi(y)\cdot\psi(y)\frac{du}{dz}. \quad . \quad . \quad . \quad . \qquad (7)$$

Applying the symbol $\frac{d}{dx}$ to equation (5), and reducing the second member by means of equation (7), we find

$$\frac{d^2u}{dx^2}=\frac{d}{dz}[\phi(y)]^2\frac{du}{dz}. \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Again, applying $\frac{d}{dx}$ to equation (8), we have

$$\frac{d^3u}{dx^3}=\frac{d}{dx}\frac{d}{dz}[\phi(y)]^2\frac{du}{dz}=\frac{d}{dz}\frac{d}{dx}[\phi(y)]^2\frac{du}{dz},$$

and, reducing by equation (7),

$$\frac{d^3u}{dx^3}=\frac{d^2}{dz^2}[\phi(y)]^3\frac{du}{dz}.$$

By successive repetitions of this process we obtain in general,

$$\frac{d^nu}{dx^n}=\frac{d^{n-1}}{dz^{n-1}}[\phi(y)]^n\frac{du}{dz}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

403. In determining the values which these derivatives assume when $x=0$, we notice that, when $x=0$, equation (1) gives $y=z$; hence $u_0=f(z)$, and from equation (4), $\left.\frac{du}{dz}\right]_{x=0}=f'(z)$. Moreover, since the differentiations indicated in the second member of equation (9) have reference only to z, we may, in this equation, assign to x its value before the differentiations are effected: therefore

$$\left.\frac{du}{dx}\right]_0=\phi(z)f'(z), \qquad \left.\frac{d^2u}{dx^2}\right]_0=\frac{d}{dz}\left\{[\phi(z)]^2f'(z)\right\},$$

$$\left.\frac{d^nu}{dx^n}\right]_0=\frac{d^{n-1}}{dz^{n-1}}\left\{[\phi(z)]^nf'(z)\right\}.$$

Substituting these values in equation (3), we obtain

$$f(y)=f(z)+x\phi(z)f'(z)+\frac{x^2}{2!}\frac{d}{dz}\left\{[\phi(z)]^2f'(z)\right\}+\cdots$$
$$+\frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left\{[\phi(z)]^nf'(z)\right\}.$$

This result is known as *Lagrange's Theorem.*

404. As an application, we expand the function y determined by

$$y=z+xe^y.$$

In this example $\phi(y) = e^y$, and $f(y) = y$, whence $f'(y) = 1$. The general term is therefore

$$\frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}e^{nz} = n^{n-1}e^{nz}\frac{x^n}{n!};$$

whence

$$y = z + e^z\cdot x + 2e^{2z}\cdot\frac{x^2}{2!} + 3^2e^{3z}\frac{x^3}{3!} + \cdots$$

To obtain the development of the function given by

$$y = 1 + xe^y,$$

we put $z = 1$ in the preceding development.

405. When the given relation between x and y is not in the form required for the application of Lagrange's theorem, an algebraic transformation sometimes enables us to make the application. Thus, if we have

$$\log y = xy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

to develop y in powers of x, we put $\log y = y'$; whence equation (1) becomes

$$y' = xe^{y'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The latter equation being in the required form (but with $z = 0$), we have

$$u = y = e^{y'}, \qquad f'(y') = e^{y'} \qquad \text{and} \qquad \phi(y') = e^{y'}.$$

Hence the general term is

$$\frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}e^{(n+1)z} = \frac{x^n}{n!}(n+1)^{n-1}e^{(n+1)z},$$

and putting $z=0$, we have

$$y=1+x+3\frac{x^2}{2!}+4^2\frac{x^3}{3!}+5^3\frac{x^4}{4!}+\cdots$$

406. Lagrange's theorem may, in fact, be applied to $f(y)$ whenever the relation between y and x is of the form

$$y=F[z+x\phi(y)];$$

for, if we put

$$t=z+x\phi(y), \qquad \text{we have} \qquad y=F(t);$$

whence

$$u=fF(t), \qquad \text{and} \qquad t=z+x\phi F(t).$$

Lagrange's theorem is therefore immediately applicable, the functions fF and ϕF taking the place of f and ϕ in the development. Hence, substituting, we have

$$f(y)=fF(z)+x\phi F(z)\frac{dfF(z)}{dz}+\frac{x^2}{2!}\frac{d}{dz}\left\{[\phi F(z)]^2\frac{dfF(z)}{dz}\right\}+\ldots$$

This form of the series is called *Laplace's Theorem.*

The example in Art. 405 may be regarded as a case of this theorem; for the given equation may be written in the form

$$y=e^{xy},$$

and we have in this case $f(y)=y$, also

$$F(t)=e^t, \qquad t=xy, \qquad z=0, \qquad \text{and} \qquad \phi(y)=y.$$

Both $fF(z)$ and $\phi F(z)$ reduce to e^z, and are identical with $f(z)$ and $\phi(z)$ of Art. 405.

Examples XXXI.

1. Change the independent variable from x to z in the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0, \qquad \text{when} \qquad x=\epsilon^z.$$

$$\frac{d^2y}{dz^2}+y=0.$$

2. Change the independent variable from x to y in the equation

$$\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2=0.$$

$$\frac{d^2x}{dy^2}-2y\frac{dx}{dy}=0.$$

3. Change the independent variable from y to x in the equation

$$(1-y^2)\frac{d^2u}{dy^2}-y\frac{du}{dy}+a^2u=0, \qquad \text{when} \qquad y=\sin x.$$

$$\frac{d^2u}{dx^2}+a^2u=0.$$

4. Change the independent variable from x to z in the equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\frac{a^2}{x^2}y=0, \qquad \text{when} \qquad x=\frac{1}{z}.$$

$$\frac{d^2y}{dz^2}+a^2y=0.$$

5. Change the independent variable from x to z in the equation

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0, \qquad \text{when} \qquad z=\log x.$$

$$\frac{d^3y}{dz^3}+y=0.$$

6. Change the independent variable from x to t in the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0, \qquad \text{when} \qquad x^2=4t.$$

$$t\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=0.$$

7. Given $x = t + t^2$, transform $\frac{d^2u}{dt^2}$ into an expression in which x is the independent variable.

$$\frac{d^2u}{dt^2} = (1 + 4x)\frac{d^2u}{dx^2} + 2\frac{du}{dx}.$$

8. Change the independent variable from x to z in the equation

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2ax = 0, \qquad \text{when } z = \log[x + \sqrt{(1 + x^2)}].$$

$$\frac{d^2y}{dz^2} + a(e^z - e^{-z}) = 0.$$

9. Change the independent variable from x to z in the equation

$$(a^2 - x^2)\frac{d^2y}{dx^2} - \frac{a^2}{x}\frac{dy}{dx} + \frac{x^3}{a} = 0, \qquad \text{when} \qquad x^2 + z^2 = a^2.$$

$$a\frac{d^2y}{dz^2} + 1 = 0.$$

10. If $x = e^\theta$, prove the symbolic identities

$$x\frac{d}{dx} = \frac{d}{d\theta}, \qquad x^{1-r}\frac{d}{dx}x^r = \frac{d}{d\theta} + r;$$

and thence, more generally,

$$x^{n-r}\frac{d^n}{dx^n}x^r \cdot u = \left[\frac{d}{d\theta} + r\right]\left[\frac{d}{d\theta} + r - 1\right]\left[\frac{d}{d\theta} + r - 2\right]\cdots$$
$$\left[\frac{d}{d\theta} + r - n + 1\right]u,$$

in which r admits of negative and fractional values.

11. Given $x = e^\theta$, transform by the result of Ex. 10 the equation

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

$$\left[\frac{d}{d\theta} - 1\right]^2\left[\frac{d}{d\theta} + 2\right]y = 0.$$

12. When $x = e^\theta$, prove by means of Ex. 10

$$x^{\frac{3}{2}}\frac{d^2}{dx^2}x^{\frac{1}{2}} \cdot u = \left[\frac{d^2}{d\theta^2} - \frac{1}{4}\right] \cdot u,$$

and verify when $u = \sin x$.

13. When $x = e^\theta$, show that

$$\frac{d^2}{dx^2}x^{-3}\frac{d}{dx}x^5\cdot u = e^{-\theta}\left[\frac{d}{d\theta}+5\right]\left[\frac{d}{d\theta}+1\right]\frac{d}{d\theta}\cdot u.$$

14. Given $x = a\,(1-\cos t)$ and $y = a\,(nt+\sin t)$, prove that

$$\frac{d^2y}{dx^2} = -\frac{n\cos t+1}{a\sin^3 t}.$$

15. Given $x = r\cos\theta$ and $y = r\sin\theta$, y being a function of x, prove that

$$\frac{d^2y}{dx^2} = \frac{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right)^3}.$$

16. Find the general value of $\frac{d^2u}{dy\,dx}$ in terms of derivatives with respect to r and θ (see Art. 395); also the special value, in the case of rectangular and polar coordinates.

$$\frac{d^2u}{dx\,dy} = \frac{du}{dr}\frac{d^2r}{dx\,dy} + \frac{du}{d\theta}\frac{d^2\theta}{dx\,dy} + \frac{d^2u}{dr^2}\frac{dr}{dx}\frac{dr}{dy}$$
$$+\frac{d^2u}{dr\,d\theta}\left[\frac{dr}{dx}\frac{d\theta}{dy} + \frac{dr}{dy}\frac{d\theta}{dx}\right] + \frac{d^2u}{d\theta^2}\frac{d\theta}{dx}\frac{d\theta}{dy};$$

$$\frac{d^2u}{dx\,dy} = -\frac{\sin\theta\cos\theta}{r}\frac{du}{dr} + \frac{\sin^2\theta-\cos^2\theta}{r^2}\frac{du}{d\theta}$$
$$+\sin\theta\cos\theta\frac{d^2u}{dr^2} + \frac{\cos^2\theta-\sin^2\theta}{r}\frac{d^2u}{dr\,d\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{d^2u}{d\theta^2}.$$

17. Given $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$x\frac{du}{dy} - y\frac{du}{dx} = \frac{du}{d\theta} \qquad \text{and that} \qquad x\frac{du}{dx} + y\frac{du}{dy} = r\frac{du}{dr}.$$

18. If $\xi = x\cos\alpha - y\sin\alpha$, and $\eta = x\sin\alpha + y\cos\alpha$, prove that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2};$$

thence show that $\nabla^2 u$ is unchanged in value by any change in the rectangular planes of reference.

19. Transform the expression $r^2\dfrac{d^2u}{dr^2} + \dfrac{d^2u}{d\theta^2}$ into a function in which x and y are the independent variables, having given $x = r\cos\theta$ and $y = r\sin\theta$.

$$r^2\frac{d^2u}{dr^2} + \frac{d^2u}{d\theta^2} = (x^2+y^2)\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right] - x\frac{du}{dx} - y\frac{du}{dy}.$$

20. If $s = e^x + e^y$, and $t = e^{-x} + e^{-y}$, prove that

$$\frac{d^2u}{dx^2} + 2\frac{d^2u}{dx\,dy} + \frac{d^2u}{dy^2} = s^2\frac{d^2u}{ds^2} - 2st\frac{d^2u}{ds\,dt} + t^2\frac{d^2u}{dt^2} + s\frac{du}{ds} + t\frac{du}{dt}.$$

21. If $x = ae^\theta\cos\phi$, and $y = ae^\theta\sin\phi$, prove that

$$v^2\frac{d^2u}{dx^2} - 2xy\frac{d^2u}{dx\,dy} + x^2\frac{d^2u}{dy^2} = \frac{d^2u}{d\phi^2} + \frac{du}{d\theta}.$$

22. Given $v = z + xe^{py}$, expand e^{my} in powers of x.

$$e^{my} = e^{mz} + me^{(p+m)z}x + m(2p+m)e^{(2p+m)z}\frac{x^2}{2!} + \cdots$$
$$+ m(np+m)^{n-1}e^{(np+m)z}\frac{x^n}{n!} + \cdots$$

23. Given $y = a + xy^3$, expand y in powers of x.

$$y = a + a^3x + 6a^5\frac{x^2}{2!} + 9\cdot 8a^7\frac{x^3}{3!} + 12\cdot 11\cdot 10\,a^9\frac{x^4}{4!} + \cdots$$
$$+ \frac{(3n)!}{(2n+1)!\,n!}\,x^n + \cdots$$

24. Given $v = z + x\dfrac{y^2 - 1}{2}$, expand y in powers of x.

$$y + \tfrac{1}{2}(z^2 - 1)x + \tfrac{1}{2}z(z^2 - 1)x^2 + \tfrac{1}{8}(5z^4 - 6z^2 + 1)x^3 + \cdots$$

25. Given $v = a + by^m$, expand y in powers of b.

$$y = a + a^m b + 2ma^{2m-1}\frac{b^2}{2!} + 3m(3m - 1)a^{3m-2}\frac{b^3}{3!} + \cdots$$

26. Given $x^5 + 4x + 2 = 0$, determine the value of x.

$$x = -\frac{1}{2} + \frac{1}{2^7} - \frac{5}{2^{13}} + \frac{35}{2^{19}} - \cdots$$

27. Given $y = a + xy^3$, expand y^3 in powers of x.

$$y^3 = a^3 + 3a^5x + 8a^7\cdot\frac{3x^2}{2!} + 11\cdot 10a^9\frac{3x^3}{3!} + 14\cdot 13\cdot 12a^{11}\frac{3x^4}{4!} + \cdots$$

28. Given $y = e + x \log y$, where e is the Napierian base, expand y in powers of x.

$$y = e + x + \frac{x^2}{e} + \frac{x^3}{2e^2} - \frac{x^4}{6e^3} + \cdots$$

29. Given $y = xe^{-y}$, expand $\sin(\alpha + y)$ in powers of x.

Solution:

The coefficient of $\dfrac{x^n}{n!}$ is $\dfrac{d^{n-1}}{dz^{n-1}}[e^{-nz}\cos(\alpha + z)]$; in which z is to be put equal to zero after the differentiations. By the method of Art. 103, we find

$$\frac{d^{n-1}}{dz^{n-1}}[e^{-nz}\cos(\alpha + z)] = (-1)^{n-1}(1 + n^2)^{\frac{n-1}{2}}\cos[\alpha + z - (n - 1)\cot^{-1}n];$$

hence

$$\sin(\alpha+y)=\sin\alpha+x\cos\alpha+\cdots$$

$$-(-1)^n(1+n^2)^{\frac{n-1}{2}}\cos\left[\alpha-(n-1)\cot^{-1}n\right]\frac{x^n}{n!}-\cdots$$

30. Develop $[1+\sqrt{(1-e^2)}]^{-p}$ in powers of e.

Put $E=1+\sqrt{(1-e^2)}$, *whence* $E=2-\frac{e^2}{E}$.

$$[1+\sqrt{(1-e^2)}]^{-p}=\frac{1}{2^p}+\frac{pe^2}{2^{p+2}}+\frac{p(p+3)e^4}{2^{p+4}\cdot 2!}+\frac{p(p+4)(p+5)e^6}{2^{p+6}\cdot 3!}+\cdots$$

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